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OPERATOR SYNTHESIS AND TENSOR PRODUCTS

G.K. ELEFThERAKIS AND I.G. TODOROV

ABSTRACT. We show that Kraus' property S_σ is preserved under taking weak* closed sums with masa-bimodules of finite width, and establish an intersection formula for weak* closed spans of tensor products, one of whose terms is a masa-bimodule of finite width. We initiate the study of the question of when operator synthesis is preserved under the formation of products and prove that the union of finitely many sets of the form $\kappa \times \lambda$, where κ is a set of finite width, while λ is operator synthetic, is, under a necessary restriction on the sets λ , again operator synthetic. We show that property S_σ is preserved under spatial Morita subordination.

1. INTRODUCTION

Operator synthesis was introduced by W.B. Arveson in his seminal paper [1] as an operator theoretic version of the notion of spectral synthesis in Harmonic Analysis, and was subsequently developed by J. Froelich, A. Katavolos, J. Ludwig, V.S. Shulman, N. Spronk, L. Turowska and the authors [8], [12], [16], [25], [26], [27], [28], [29], among others. It was shown in [12], [27] and [20] that, for a large class of locally compact groups G , given a closed subset E of G , there is a canonical way to produce a subset E^* of the direct product $G \times G$, so that the set E satisfies spectral synthesis if and only if the set E^* satisfies operator synthesis. Thus, the well-known, and still open, problem of whether the union of two sets of spectral synthesis satisfies spectral synthesis can be viewed as a special case of the problem asking whether the union of two operator synthetic sets is operator synthetic.

Another problem in Harmonic Analysis asks when the product of two sets of spectral synthesis is again synthetic. The analogous question in the operator theory setting is closely related to *property* S_σ , introduced by J. Kraus in [17]. It is widely recognised that functional analytic tensor products display a larger degree of subtlety than the algebraic ones, the reason for this being the fact that they are defined as the *completion* of the *algebraic* tensor product of two objects (say, operator algebras, or operator spaces) with respect to an appropriate topology. Therefore, it is usually not an easy task to determine the intersection of two spaces, both given as completed tensor products. Such issues give rise to a number of important concepts in Operator Algebra Theory, *e.g.* exactness [23]. Property S_σ is instrumental in describing such intersections, and is closely related to a number of important approximation properties. In particular, it was shown in [18] to be equivalent to the σ -weak approximation property, while in [13], an equivalence of

when a group von Neumann algebra $VN(G)$ possesses property S_σ was formulated in terms of an approximation property of the underlying group G .

In this paper, we initiate the study of the question of when the direct product of two operator synthetic sets is operator synthetic. Furthermore, we combine the directions of investigation highlighted in the previous two paragraphs by studying the question of when the union of direct products of operator synthetic sets is operator synthetic. The setting of operator synthesis is provided by the theory of masa-bimodules (see [1], [9], [25] and [26]). A prominent role in our study is played by the masa-bimodules of *finite width*. This class is a natural extension of the class of CSL algebras of finite width, which was introduced in [1] as a far reaching, yet tractable, generalisation of *nest algebras* [4]. It was shown in [14] that CSL algebras of finite width possess property S_σ . However, this class has for long remained the main example of operator spaces known to have this property. We note that the question of whether every weak* closed masa-bimodule possesses property S_σ is still open (see [18]).

It should be noted that masa-bimodules of finite width have been studied in a number of other contexts. They include as a subclass the masa-bimodules which are *ternary rings of operators* [29], a class of operator spaces that has been studied extensively for the purposes of Operator Space Theory [3]. The supports of masa-bimodules of finite width (called henceforth *sets of finite width*) are precisely the sets of solutions of systems of inequalities, and were shown in [25] and [28] to be operator synthetic, providing in this way the largest single class of sets that are known to satisfy operator synthesis. It was shown in [8] that the union of an operator synthetic set and a set of finite width is operator synthetic. In [24], this line of investigation was continued by showing that masa-bimodules of finite width satisfy a rank one approximation property, and a large class of examples of *sets of operator multiplicity* was exhibited within this class. They were the motivating example for the introduction and study of \mathfrak{J} -decomposable masa-bimodules in [8].

The weak* closed masa-bimodules are precisely the weak* closed invariant subspaces of *Schur multipliers* or, equivalently, of weak* continuous (completely) bounded masa-bimodule maps. The projections in the algebra of all Schur multipliers, called henceforth *Schur idempotents*, were at the core of the methods developed in [8] in order to address the union problem, as well as the closely related problem of the reflexivity of weak* closed spans.

Here we significantly extend the techniques whose development was initiated in [8] by establishing an intersection formula involving tensor products and applying it to the study of the product and union problems described above. Simultaneously, we initiate the study of the question of whether property S_σ is preserved under taking weak* closed spans.

The paper is organised as follows. After gathering some preliminary notions and results in Section 2, we address in Section 3 the preservation problem for property S_σ outlined in the previous paragraph, showing that the class of spaces possessing S_σ is closed under taking weak* closed sums with masa-bimodules of finite width (Theorem 3.7). As a consequence, the weak* closed span of any finite number of masa-bimodules of finite width possesses property S_σ .

In Section 4, we establish the intersection formula

$$(1) \quad \bigcap_{j_1, \dots, j_r} \overline{\mathcal{B}_{j_1}^1 \otimes \mathcal{U}_1 + \dots + \mathcal{B}_{j_r}^r \otimes \mathcal{U}_r} = \overline{(\cap_{j_1} \mathcal{B}_{j_1}^1) \otimes \mathcal{U}_1 + \dots + (\cap_{j_r} \mathcal{B}_{j_r}^1) \otimes \mathcal{U}_r},$$

valid for all masa-bimodules $\mathcal{B}_{j_p}^p$ of finite width and all weak* closed spaces of operators \mathcal{U}_p , $p = 1, \dots, r$, $j_p = 1, \dots, m_p$ (Corollary 4.21). In Section 5, we formalise the relation between property S_σ and the problem for the synthesis of products (see Corollary 5.4). As part of Proposition 5.3, we establish a subspace version of the relation between *Fubini products* and the *algebra tensor product formula* discussed in [17]. These results, along with the formula (1), are used to show that the union of finitely many products $\kappa_i \times \lambda_i$, where the sets κ_i are of finite width, while λ_i are operator synthetic sets satisfying certain necessary restrictions, is operator synthetic (Theorem 5.9).

In Section 6, we show that property S_σ is preserved under spatial Morita subordination. As a corollary, we obtain that if \mathcal{L}_1 and \mathcal{L}_2 are isomorphic CSL's then the CSL algebra $\text{Alg}\mathcal{L}_1$ possesses property S_σ if and only if $\text{Alg}\mathcal{L}_2$ does so.

It is natural to wonder whether our results are valid for the more general class consisting of intersections of \mathfrak{I} -decomposable spaces introduced in [8]. We note that this class contains properly the class of masa-bimodules of finite width. Progress in this direction would rely on the answer of the question of whether the *approximately \mathfrak{I} -injective* masa-bimodules (that is, the intersections of descending sequences of ranges of uniformly bounded Schur idempotents) satisfy property S_σ ; this question, however, is still open.

Finally, we wish to point out that our results are stated for bimodules over one masa. This setting is not less general than the one where bimodules over two, possibly distinct, masas are considered; indeed, a standard two by two matrix argument, discussed in detail in Section 2, allows us to reduce the latter case to the former.

2. PRELIMINARIES

In this section, we collect some preliminary notions and results that will be needed in the sequel. Throughout the paper, H and K will denote Hilbert spaces. We let $\mathcal{B}(H, K)$ be the space of all bounded linear operators from H into K , and write $\mathcal{B}(H) = \mathcal{B}(H, H)$. The space $\mathcal{B}(H, K)$ is the dual of the ideal of all trace class operators from K into H , and can hence be endowed with a weak* topology; we note that this is the weakest topology on $\mathcal{B}(H, K)$ with respect to which the functionals ω of the form

$$\omega(T) = \sum_{k=1}^{\infty} (T\xi_k, \eta_k), \quad T \in \mathcal{B}(H, K),$$

where $(\xi_k)_{k \in \mathbb{N}} \subseteq H$ and $(\eta_k)_{k \in \mathbb{N}} \subseteq K$ are square summable sequences of vectors, are continuous. In the sequel, we denote by $\overline{\mathcal{U}}$ the weak* closure of a set $\mathcal{U} \subseteq \mathcal{B}(H, K)$.

Let $\mathcal{V} \subseteq \mathcal{B}(H)$ and $\mathcal{U} \subseteq \mathcal{B}(K)$ be weak* closed subspaces. We denote by $\mathcal{V} \overline{\otimes} \mathcal{U}$ the weak* closed subspace of $\mathcal{B}(H \otimes K)$ generated by the operators of the form

$S \otimes T$, where $S \in \mathcal{V}$ and $T \in \mathcal{U}$. Here, $H \otimes K$ is the Hilbertian tensor product of H and K , and we use the natural identification

$$\mathcal{B}(H \otimes K) \equiv \mathcal{B}(H) \bar{\otimes} \mathcal{B}(K).$$

We will use some basic notions from Operator Space Theory; we refer the reader to the monographs [3], [5], [22] and [23] for the relevant definitions. If \mathcal{X} is a linear space, we denote by id the identity map on \mathcal{X} . The range of a linear map ϕ on \mathcal{X} is denoted by $\text{Ran } \phi$. As customary, the map ϕ is called idempotent if $\phi \circ \phi = \phi$; we let $\phi^\perp = \text{id} - \phi$. If \mathcal{X}_1 and \mathcal{X}_2 are subspaces of \mathcal{X} , we set $\mathcal{X}_1 + \mathcal{X}_2 = \{x_1 + x_2 : x_i \in \mathcal{X}_i, i = 1, 2\}$. If $\mathcal{V}_i \subseteq \mathcal{B}(H)$ and $\mathcal{U}_i \subseteq \mathcal{B}(K)$ are weak* closed subspaces, $i = 1, 2$, and $\phi : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ and $\psi : \mathcal{U}_1 \rightarrow \mathcal{U}_2$ are completely bounded weak* continuous maps, then there exists a (unique) completely bounded weak* continuous map $\phi \otimes \psi : \mathcal{V}_1 \bar{\otimes} \mathcal{U}_1 \rightarrow \mathcal{V}_2 \bar{\otimes} \mathcal{U}_2$ such that $\phi \otimes \psi(A \otimes B) = \phi(A) \otimes \psi(B)$, $A \in \mathcal{V}_1$, $B \in \mathcal{U}_1$ [3]. In the case $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{B}(K)$, we write throughout the paper $\tilde{\phi} = \phi \otimes \text{id}$. We denote by \mathcal{V}_* the space of all weak* continuous functionals on \mathcal{V} . If $\mathcal{V} \subseteq \mathcal{B}(H)$ is a weak* closed subspace of operators and $\omega \in \mathcal{V}_*$ then we set $R_\omega = \tilde{\omega}$; thus, $R_\omega : \mathcal{V} \bar{\otimes} \mathcal{B}(K) \rightarrow \mathcal{B}(K)$ is the *Tomiyama's right slice map* corresponding to ω (here we have use the natural identification $\mathbb{C} \bar{\otimes} \mathcal{B}(K) \equiv \mathcal{B}(K)$). We note that $R_\omega(A \otimes B) = \omega(A)B$, $A \in \mathcal{V}$, $B \in \mathcal{B}(K)$. If, further, $\mathcal{U} \subseteq \mathcal{B}(K)$ is a weak* closed subspace, the *Fubini product* $\mathcal{F}(\mathcal{V}, \mathcal{U})$ of \mathcal{V} and \mathcal{U} is the subspace of $\mathcal{V} \bar{\otimes} \mathcal{B}(K)$ given by

$$\mathcal{F}(\mathcal{V}, \mathcal{U}) = \{T \in \mathcal{V} \bar{\otimes} \mathcal{B}(K) : R_\omega(T) \in \mathcal{U}, \text{ for all } \omega \in \mathcal{V}_*\}.$$

If $\xi \in H_1$ and $\eta \in H_2$, we let $\omega_{\xi, \eta}$ be the vector functional on $\mathcal{B}(H)$ given by $\omega_{\xi, \eta}(A) = (A\xi, \eta)$, $A \in \mathcal{B}(H)$; we use the same symbol to denote the restriction of $\omega_{\xi, \eta}$ to the subspace $\mathcal{U} \subseteq \mathcal{B}(H)$.

It is easy to notice that $\mathcal{V} \bar{\otimes} \mathcal{U} \subseteq \mathcal{F}(\mathcal{V}, \mathcal{U})$. The subspace \mathcal{V} is said to possess *property S_σ* if $\mathcal{F}(\mathcal{V}, \mathcal{U}) = \mathcal{V} \bar{\otimes} \mathcal{U}$ for all weak* subspaces $\mathcal{U} \subseteq \mathcal{B}(K)$ and all Hilbert spaces K . This notion was introduced by Kraus in [17], where he showed that $\mathcal{B}(K)$ possesses property S_σ . From this fact, one can easily derive the formula

$$\mathcal{F}(\mathcal{V}, \mathcal{U}) = (\mathcal{V} \bar{\otimes} \mathcal{B}(K)) \cap (\mathcal{B}(H) \bar{\otimes} \mathcal{U}).$$

Now suppose that H is a separable Hilbert space and $\mathcal{D} \subseteq \mathcal{B}(H)$ is a maximal abelian selfadjoint algebras (for brevity, masa). A linear map ϕ on $\mathcal{B}(H)$ is called *\mathcal{D} -bimodular*, or a *masa-bimodule map* when \mathcal{D} is clear from the context, provided $\phi(BXA) = B\phi(X)A$, for all $X \in \mathcal{B}(H)$, $A, B \in \mathcal{D}$. We call the completely bounded weak* continuous \mathcal{D} -bimodular maps on $\mathcal{B}(H)$ *Schur maps* (relative to the masas \mathcal{D}); a Schur map that is also an idempotent is called a *Schur idempotent*. This terminology is natural in view of the fact that Schur maps correspond precisely to Schur multipliers, provided a particular coordinate representation of \mathcal{D} is chosen. We refer the reader to [8] for details; we will return to this perspective in Section 5.

A *\mathcal{D} -bimodule*, or simply a *masa-bimodule* when \mathcal{D} is understood from the context, is a subspace $\mathcal{V} \subseteq \mathcal{B}(H)$ such that $BXA \in \mathcal{V}$ whenever $X \in \mathcal{V}$, $A, B \in \mathcal{D}$. Masa-bimodules will be assumed to be weak* closed throughout the paper; they are precisely the weak* closed subspaces invariant under all Schur maps (see [8,

Proposition 3.2]). A weak* closed masa-bimodule \mathcal{M} is called *ternary* [16], [29] if \mathcal{M} is a ternary ring of operators, that is, if $TS^*R \in \mathcal{M}$ whenever $T, S, R \in \mathcal{M}$ (see also [3]). It is not difficult to see that every ternary masa-bimodule is the intersection of a descending sequence of ranges of contractive Schur idempotents; this fact will be used extensively hereafter. It is easy to notice that the ternary masa-bimodules acting on a single Hilbert space which are unital algebras are precisely the von Neumann algebras with abelian commutant.

We note that masa-bimodules are usually defined as subspaces $\mathcal{V} \subseteq \mathcal{B}(H_1, H_2)$, where H_1 and H_2 are two (possibly distinct) Hilbert spaces, such that $BXA \in \mathcal{V}$ whenever $X \in \mathcal{V}$, $A \in \mathcal{D}_1$ and $B \in \mathcal{D}_2$, with $\mathcal{D}_1 \subseteq \mathcal{B}(H_1)$ and $\mathcal{D}_2 \subseteq \mathcal{B}(H_2)$ being two (possibly distinct) masas. However, this seemingly more general situation can be recovered from the one described in the previous paragraph, using a standard two by two matrix trick. Namely, let $H = H_2 \oplus H_1$, $\mathcal{D} = \mathcal{D}_2 \oplus \mathcal{D}_1$ and $\tilde{\mathcal{V}} \subseteq \mathcal{B}(H)$ be given by

$$(2) \quad \tilde{\mathcal{V}} = \left\{ \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} : T \in \mathcal{V} \right\}.$$

Then \mathcal{D} is a masa, and $\tilde{\mathcal{V}}$ is a \mathcal{D} -bimodule. Moreover, $\tilde{\mathcal{V}}$ is weak* closed if and only if \mathcal{V} is weak* closed, and $\tilde{\mathcal{V}}$ is ternary if and only if \mathcal{V} is ternary.

A *nest* on a Hilbert space H is a totally ordered family of closed subspaces of H that contains the intersection and the closed linear span (denoted \vee) of any of its subsets. A *nest algebra* is the subalgebra of $\mathcal{B}(H)$ of all operators leaving invariant each subspace of a given nest. A *nest algebra bimodule* is a subspace $\mathcal{V} \subseteq \mathcal{B}(H)$ for which there exists a nest algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ such that $\mathcal{A}\mathcal{V}\mathcal{A} \subseteq \mathcal{V}$. All nest algebra bimodules will be assumed to be weak* closed. A \mathcal{D} -bimodule \mathcal{V} is said to have *finite width* if it is of the form $\mathcal{V} = \mathcal{V}_1 \cap \cdots \cap \mathcal{V}_k$, where each \mathcal{V}_j is a \mathcal{D} -bimodule that is also a nest algebra bimodule. The smallest k with this property is called the *width* of \mathcal{V} . It will be convenient to define the width of $\mathcal{B}(H)$ to be zero. We note that every ternary masa-bimodule has width at most two [16]. If each \mathcal{V}_j is a nest algebra then \mathcal{V} is called a CSL algebra of finite width [1]. It was shown in [17] that von Neumann algebras with abelian commutant possess property S_σ and in [14] that every CSL algebra of finite width possesses property S_σ .

We note that nest algebra bimodules can also be defined as subspaces $\mathcal{V} \subseteq \mathcal{B}(H_1, H_2)$, where H_1 and H_2 are two (possibly distinct) Hilbert spaces, such that $BXA \in \mathcal{V}$ whenever $X \in \mathcal{V}$, $A \in \mathcal{A}_1$ and $B \in \mathcal{A}_2$, with $\mathcal{A}_1 \subseteq \mathcal{B}(H_1)$ and $\mathcal{A}_2 \subseteq \mathcal{B}(H_2)$ being two (possibly distinct) nest algebras. Letting \mathcal{N}_1 (resp. \mathcal{N}_2) be the nest of \mathcal{A}_1 (resp. \mathcal{A}_2), $H = H_2 \oplus H_1$,

$$\mathcal{N} = \{N_2 \oplus 0, I \oplus N_1 : N_1 \in \mathcal{N}_1, N_2 \in \mathcal{N}_2\}$$

and \mathcal{A} be the nest algebra leaving all elements of \mathcal{N} invariant, one can easily see that the space defined in (2) is an \mathcal{A} -bimodule.

Suppose that $\mathcal{V} \subseteq \mathcal{B}(H)$ is a nest algebra bimodule. It was shown in [10] that there exists a nest $\mathcal{N} \subseteq \mathcal{B}(H)$ and an increasing \vee -preserving map $\varphi : \mathcal{N} \rightarrow \mathcal{N}$ such that

$$\mathcal{V} = \{X \in \mathcal{B}(H) : XN = \varphi(N)XN, N \in \mathcal{N}\}.$$

Let $\{P_i\}_{i \in \mathbb{N}} \subseteq \mathcal{N}$ be a (countable) subset such that $\{P_i \oplus \varphi(P_i) : i \in \mathbb{N}\}$ is dense in $\{P \oplus \varphi(P) : P \in \mathcal{N}\}$ in the strong operator topology, and

$$\mathcal{F}_n = \{0, P_1, P_2, \dots, P_n, I\} = \{0 < N_1 < N_2 < \dots < N_n < I\}.$$

Set $N_0 = 0$ and $N_{n+1} = I$ and let $\phi_n, \psi_n : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ be the Schur idempotents (relative to any masa \mathcal{D} with $\mathcal{N} \subseteq \mathcal{D}$) given by

$$\begin{aligned} \phi_n(X) &= \sum_{i=0}^n (\varphi(N_{i+1}) - \varphi(N_i))X(N_{i+1} - N_i), \quad X \in \mathcal{B}(H), \\ \psi_n(X) &= \sum_{0 \leq i < j \leq n} (\varphi(N_{i+1}) - \varphi(N_i))X(N_{j+1} - N_j), \quad X \in \mathcal{B}(H), \end{aligned}$$

and \mathcal{M}_n and \mathcal{W}_n be the ranges of ϕ_n and ψ_n , respectively. We have that

$$\phi_n \psi_n = 0, \quad \mathcal{W}_n \subseteq \mathcal{V} \subseteq \mathcal{W}_n + \mathcal{M}_n, \quad \mathcal{W}_n \subseteq \mathcal{W}_{n+1}, \quad \mathcal{M}_{n+1} \subseteq \mathcal{M}_n, \quad n \in \mathbb{N},$$

and $\cap_{n=1}^{\infty} \mathcal{M}_n \subseteq \mathcal{V}$. We will call the family $(\phi_n, \psi_n, \mathcal{M}_n, \mathcal{W}_n)_{n \in \mathbb{N}}$ a *decomposition scheme* for \mathcal{V} . Decomposition schemes were first explicitly used (although not referred to as such) in [8] for the study of reflexivity and synthesis problems. We note that, if $\psi_{n,p}$, $p = 1, \dots, n$, is given by

$$\psi_{n,p}(X) = \sum_{j-i=p} (\varphi(N_{i+1}) - \varphi(N_i))X(N_{j+1} - N_j), \quad X \in \mathcal{B}(H),$$

then $\psi_n = \sum_{p=1}^n \psi_{n,p}$, $\psi_{n,p} \psi_{n,q} = 0$ if $p \neq q$ and $\|\psi_{n,p}\| = 1$, $p = 1, \dots, n$.

3. STABILITY UNDER SUMMATION WITH MODULES OF FINITE WIDTH

The main results in this section are Theorem 3.7 and the associated Corollary 3.8, which show that tensor product formulas are preserved under taking weak* closed sums with masa-bimodules of finite width. We begin with some lemmas which will be used in this and the subsequent section.

Lemma 3.1. *Let ϕ be a weak* continuous completely bounded linear map on $\mathcal{B}(H)$ and $\mathcal{V}, \mathcal{V}_i \subseteq \mathcal{B}(H)$, $\mathcal{U}, \mathcal{U}_i \subseteq \mathcal{B}(K)$ be weak* closed subspaces, $i = 1, \dots, n$. Suppose that \mathcal{V} is invariant under ϕ .*

(i) *We have $\tilde{\phi}(\overline{\sum_{i=1}^n \mathcal{V}_i \bar{\otimes} \mathcal{U}_i}) \subseteq \overline{\sum_{i=1}^n \phi(\mathcal{V}_i) \bar{\otimes} \mathcal{U}_i}$. In particular, $\tilde{\phi}$ leaves $\mathcal{V} \bar{\otimes} \mathcal{U}$ invariant.*

(ii) *If ϕ is an idempotent then*

$$(\text{Ran } \phi \bar{\otimes} \mathcal{U}) \cap (\mathcal{V} \bar{\otimes} \mathcal{U}) = \phi(\mathcal{V}) \bar{\otimes} \mathcal{U} = \tilde{\phi}(\mathcal{V} \bar{\otimes} \mathcal{U}).$$

In particular, ranges of Schur idempotents possess property S_σ .

Proof. (i) Fix $i \in \{1, \dots, n\}$ and suppose that $T \in \mathcal{V}_i \bar{\otimes} \mathcal{U}_i$. Then T can be approximated in the weak* topology by operators of the form $\sum_{j=1}^k A_j \otimes B_j$, where $A_j \in \mathcal{V}_i$, $B_j \in \mathcal{U}_i$, $j = 1, \dots, k$; therefore, $\tilde{\phi}(T)$ can be approximated in the weak* topology by operators of the form $\sum_{j=1}^k \phi(A_j) \otimes B_j$, where $A_j \in \mathcal{V}_i$, $B_j \in \mathcal{U}_i$, $j = 1, \dots, k$. Hence, $\tilde{\phi}(T) \in \overline{\phi(\mathcal{V}_i) \bar{\otimes} \mathcal{U}_i}$. The conclusion now follows from the linearity and the weak* continuity of $\tilde{\phi}$.

(ii) Since ϕ is an idempotent, $\phi(\mathcal{V})$ is weak* closed. By (i), $\tilde{\phi}(\mathcal{V} \bar{\otimes} \mathcal{U}) \subseteq \phi(\mathcal{V}) \bar{\otimes} \mathcal{U}$ while, since $\phi(\mathcal{V}) \subseteq \mathcal{V}$, we have $\phi(\mathcal{V}) \bar{\otimes} \mathcal{U} \subseteq (\text{Ran } \phi \bar{\otimes} \mathcal{U}) \cap (\mathcal{V} \bar{\otimes} \mathcal{U})$. Suppose that $T \in (\text{Ran } \phi \bar{\otimes} \mathcal{U}) \cap (\mathcal{V} \bar{\otimes} \mathcal{U})$. Then, by (i), $\tilde{\phi}^\perp(T) = 0$ and hence

$$T = \tilde{\phi}(T) \in \tilde{\phi}(\mathcal{V} \bar{\otimes} \mathcal{U}).$$

Finally, if

$$T \in (\text{Ran } \phi \bar{\otimes} \mathcal{B}(K)) \cap (\mathcal{B}(H) \bar{\otimes} \mathcal{U})$$

then, by the previous paragraph, $T = \tilde{\phi}(T) \in \phi(\mathcal{B}(H)) \bar{\otimes} \mathcal{U}$; thus, $\text{Ran } \phi$ possesses S_σ . \square

Lemma 3.2. *Let ϕ be a weak* continuous completely bounded idempotent acting on $\mathcal{B}(H)$, $\mathcal{V} \subseteq \mathcal{B}(H)$ be a weak* closed subspace invariant under ϕ , $\mathcal{U} \subseteq \mathcal{B}(K)$ be a weak* closed subspace and $\mathcal{W} \subseteq \mathcal{B}(H \otimes K)$ be a weak* closed subspace invariant under $\tilde{\phi}$. Then*

$$\overline{\mathcal{V} \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{\text{Ran } \phi \bar{\otimes} \mathcal{U} + \mathcal{W}} = \overline{(\text{Ran } \phi \cap \mathcal{V}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

Proof. Suppose

$$T \in \overline{\mathcal{V} \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{\text{Ran } \phi \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

By Lemma 3.1 and the invariance of \mathcal{W} under $\tilde{\phi}$, we have that $\tilde{\phi}^\perp(T) \in \mathcal{W}$; similarly,

$$\tilde{\phi}(T) \in \overline{\phi(\mathcal{V}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

It follows that

$$T = \tilde{\phi}^\perp(T) + \tilde{\phi}(T) \in \overline{(\text{Ran } \phi \cap \mathcal{V}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

The converse inclusion is trivial. \square

Lemma 3.3. *Let $\mathcal{V} \subseteq \mathcal{B}(H)$ (resp. $\mathcal{U} \subseteq \mathcal{B}(K)$) be a weak* closed subspace and ϕ (resp. ψ) be a weak* continuous completely bounded map on $\mathcal{B}(H)$ (resp. $\mathcal{B}(K)$). Then*

$$(\phi \otimes \psi)(\mathcal{F}(\mathcal{V}, \mathcal{U})) \subseteq \mathcal{F}(\overline{\phi(\mathcal{V})}, \overline{\psi(\mathcal{U})}).$$

Moreover, if ϕ and ψ are idempotents that leave \mathcal{V} and \mathcal{U} , respectively, invariant, then

$$(\phi \otimes \psi)(\mathcal{F}(\mathcal{V}, \mathcal{U})) = \mathcal{F}(\phi(\mathcal{V}), \psi(\mathcal{U})).$$

Proof. By Lemma 3.1,

$$\begin{aligned} (\phi \otimes \psi)(\mathcal{V} \bar{\otimes} \mathcal{B}(K)) &= \tilde{\phi} \circ (\text{id} \otimes \psi)(\mathcal{V} \bar{\otimes} \mathcal{B}(K)) \\ &\subseteq \tilde{\phi}(\mathcal{V} \bar{\otimes} \mathcal{B}(K)) \subseteq \overline{\phi(\mathcal{V})} \bar{\otimes} \mathcal{B}(K); \end{aligned}$$

similarly,

$$(\phi \otimes \psi)(\mathcal{B}(H) \bar{\otimes} \mathcal{U}) \subseteq \mathcal{B}(H) \bar{\otimes} \overline{\psi(\mathcal{U})}.$$

Hence,

$$\begin{aligned} (\phi \otimes \psi)(\mathcal{F}(\mathcal{V}, \mathcal{U})) &= (\phi \otimes \psi)((\mathcal{V} \bar{\otimes} \mathcal{B}(K)) \cap (\mathcal{B}(H) \bar{\otimes} \mathcal{U})) \\ &\subseteq \overline{\phi(\mathcal{V})} \bar{\otimes} \mathcal{B}(K) \cap \mathcal{B}(H) \bar{\otimes} \overline{\psi(\mathcal{U})} = \mathcal{F}(\overline{\phi(\mathcal{V})}, \overline{\psi(\mathcal{U})}). \end{aligned}$$

Now suppose that ϕ and ψ are idempotents that leave \mathcal{V} and \mathcal{U} , respectively, invariant. Then, clearly, $\mathcal{F}(\phi(\mathcal{V}), \psi(\mathcal{U})) \subseteq \mathcal{F}(\mathcal{V}, \mathcal{U})$. On the other hand, if

$$T \in \mathcal{F}(\phi(\mathcal{V}), \psi(\mathcal{U})) = (\phi(\mathcal{U}) \bar{\otimes} \mathcal{B}(K)) \cap (\mathcal{B}(H) \bar{\otimes} \psi(\mathcal{V}))$$

then

$$\phi \otimes \psi(T) = (\text{id} \otimes \psi)(\phi \otimes \text{id})(T) = T,$$

and hence $T = \phi \otimes \psi(T) \in (\phi \otimes \psi)(\mathcal{F}(\mathcal{V}, \mathcal{U}))$. Thus, $\mathcal{F}(\phi(\mathcal{V}), \psi(\mathcal{U})) \subseteq \phi \otimes \psi(\mathcal{F}(\mathcal{V}, \mathcal{U}))$; the converse inclusion follows from the previous paragraph. \square

Lemma 3.4. *Every ternary masa-bimodule possesses property S_σ .*

Proof. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a ternary masa-bimodule, \mathcal{A} be the weak* closed subspace generated by $\mathcal{M}^* \mathcal{M}$ and \mathcal{B} be the weak* closed subspace generated by $\mathcal{M} \mathcal{M}^*$. Then \mathcal{A} and \mathcal{B} are weak* closed selfadjoint algebras and $\mathcal{B} \mathcal{M} \mathcal{A} \subseteq \mathcal{M}$. Moreover, if

$$\mathcal{C} = \left\{ \begin{pmatrix} B & X \\ Y^* & A \end{pmatrix} : A \in \mathcal{A}, B \in \mathcal{B}, X, Y \in \mathcal{M} \right\},$$

then \mathcal{C} is a weak* closed selfadjoint subalgebra of $\mathcal{B}(H \oplus H)$. Let \mathcal{C}_0 be the restriction of \mathcal{C} to the subspace $\overline{\mathcal{C}(H \oplus H)}$. Then \mathcal{C}_0 is a von Neumann algebra with abelian commutant. By [17, Theorem 1.9], \mathcal{C}_0 has property S_σ . On the other hand, $\mathcal{M} = Q \mathcal{C}_0 P$, where Q and P are projections in \mathcal{C}_0 , and now [17, Proposition 1.10] implies that \mathcal{M} has property S_σ . \square

Lemma 3.5. *Let $(\mathcal{M}_n)_{n \in \mathbb{N}}$ be a descending sequence of ternary masa-bimodules in $\mathcal{B}(H)$ and $\mathcal{M} = \cap_{n \in \mathbb{N}} \mathcal{M}_n$. If $\mathcal{U} \subseteq \mathcal{B}(K)$ is a weak* closed subspace then $\cap_{n \in \mathbb{N}} (\mathcal{M}_n \bar{\otimes} \mathcal{U}) = \mathcal{M} \bar{\otimes} \mathcal{U}$.*

Proof. The inclusion $\mathcal{M} \bar{\otimes} \mathcal{U} \subseteq \cap_{n \in \mathbb{N}} (\mathcal{M}_n \bar{\otimes} \mathcal{U})$ is trivial. Suppose that $T \in \mathcal{M}_n \bar{\otimes} \mathcal{U}$ for each n . Then $L_\tau(T) \in \mathcal{M}_n$ for all n , and so $L_\tau(T) \in \mathcal{M}$, for all $\tau \in \mathcal{B}(K)_*$. On the other hand, $R_\omega(T) \in \mathcal{U}$ for all $\omega \in \mathcal{B}(H)_*$. It follows that $T \in \mathcal{F}(\mathcal{M}, \mathcal{U})$. By Lemma 3.4, $T \in \mathcal{M} \bar{\otimes} \mathcal{U}$. \square

We fix a masa $\mathcal{D} \subseteq \mathcal{B}(H)$. All Schur idempotents we consider are relative to \mathcal{D} and act on $\mathcal{B}(H)$. We will say that a sequence $(\psi_n)_{n \in \mathbb{N}}$ of Schur idempotents is *nested* if $\text{Ran } \psi_{n+1} \subseteq \text{Ran } \psi_n$ for all $n \in \mathbb{N}$.

A number of proofs below will use the following idea: Let Ω be a weak* closed subspace of operators and $(\rho_n)_{n \in \mathbb{N}}$ be a nested sequence of contractive idempotents with $\cap_{n=1}^\infty \text{Ran } \rho_n \subseteq \Omega$. In order to prove that a certain operator T belongs to Ω , it suffices to show that $\rho_n^\perp(T) \in \Omega$ for each $n \in \mathbb{N}$. Indeed, letting S be a weak* cluster point of the sequence $(\rho_n(T))_{n \in \mathbb{N}}$, we have that $S \in \cap_{n=1}^\infty \text{Ran } \rho_n \subseteq \Omega$. On the other hand, the identities $T = \rho_n(T) + \rho_n^\perp(T)$, $n \in \mathbb{N}$, show that $T - S$ is a weak* cluster point of the sequence $(\rho_n^\perp(T))_{n \in \mathbb{N}}$, and hence it belongs to Ω ; therefore, $T = S + (T - S) \in \Omega$.

Lemma 3.6. *Let $(\phi_k)_{k \in \mathbb{N}}$ be a nested sequence of contractive Schur idempotents, $\mathcal{M}_k = \text{Ran } \phi_k$, $\mathcal{M} = \cap_{k \in \mathbb{N}} \mathcal{M}_k$, ψ be a Schur idempotent with range \mathcal{Z} , $\mathcal{B} \subseteq \mathcal{B}(H)$*

be a masa-bimodule of finite width and $\mathcal{U} \subseteq \mathcal{B}(K)$ be a weak* closed subspace. Set $\mathcal{W} = \mathcal{F}(\mathcal{V}, \mathcal{U})$. Then

$$(3) \quad \cap_{k \in \mathbb{N}} \overline{(\mathcal{B} \cap \mathcal{Z} \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}} = \overline{(\mathcal{B} \cap \mathcal{Z} \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

Proof. For the proof of the lemma, it will be convenient, along with identity (3), to consider the following identity

$$(4) \quad \cap_{k \in \mathbb{N}} \overline{(\mathcal{B} \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}} = \overline{(\mathcal{B} \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

(Note that (4) is a special case of (3); indeed, take $\mathcal{Z} = \mathcal{B}(H)$). Note that the right hand sides of (3) and (4) are trivially contained in the respective left hand sides, and that the subspace \mathcal{W} is invariant under the maps of the form $\tilde{\phi}$, where ϕ is a Schur idempotent on $\mathcal{B}(H)$. The proof uses induction on the width l of \mathcal{B} . Suppose first that the width of \mathcal{B} is zero, in which case we will assume that $\mathcal{B} = \mathcal{B}(H)$. We show (3). Let

$$T \in \cap_{k \in \mathbb{N}} \overline{(\mathcal{Z} \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}}, \quad k \in \mathbb{N}.$$

Using Lemmas 3.1 and 3.3, we have

$$T \in \overline{\mathcal{M}_k \bar{\otimes} \mathcal{U} + \mathcal{F}(\mathcal{M}_k, \mathcal{U})} = \mathcal{M}_k \bar{\otimes} \mathcal{U}, \quad k \in \mathbb{N}.$$

By Lemma 3.5, $T \in \mathcal{M} \bar{\otimes} \mathcal{U}$. Now, $\tilde{\psi}^\perp(T) \in \mathcal{W}$, while $\tilde{\psi}(T) \in (\mathcal{Z} \cap \mathcal{M}) \bar{\otimes} \mathcal{U}$. It follows that

$$T = \tilde{\psi}(T) + \tilde{\psi}^\perp(T) \in \overline{(\mathcal{Z} \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

Next we show (4) for $l = 1$. In this case, \mathcal{B} is a nest algebra bimodule. Let $(\rho_n, \psi_n, \mathcal{N}_n, \mathcal{Z}_n)_{n \in \mathbb{N}}$ be a decomposition scheme for \mathcal{B} . Set $\theta_n = \text{id} - (\rho_n + \psi_n)$ and fix

$$T \in \cap_{k \in \mathbb{N}} \overline{(\mathcal{B} \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

Then $\tilde{\theta}_n(T) \in \mathcal{W}$ for each n . Also,

$$\tilde{\psi}_n(T) \in \cap_{k \in \mathbb{N}} \overline{(\mathcal{Z}_n \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}}$$

and hence, by the previous paragraph,

$$\tilde{\psi}_n(T) \in \overline{(\mathcal{Z}_n \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \overline{(\mathcal{B} \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

Suppose that S is a weak* cluster point of the sequence $(\tilde{\rho}_n(T))_{n \in \mathbb{N}}$. Then, again by the previous paragraph,

$$\begin{aligned} S &\in \cap_{k \in \mathbb{N}} \overline{(\mathcal{N}_k \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \overline{((\cap_{k \in \mathbb{N}} \mathcal{N}_k) \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}} \\ &\subseteq \overline{(\mathcal{B} \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}}. \end{aligned}$$

It follows that $T \in \overline{(\mathcal{B} \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}}$.

Now suppose that (3) holds if the width of \mathcal{B} is at most $l - 1$, while (4) holds if the width of \mathcal{B} is at most l . For a nest algebra bimodule \mathcal{A} , fix a decomposition scheme $(\rho_n, \psi_n, \mathcal{N}_n, \mathcal{Z}_n)_{n \in \mathbb{N}}$ and set $\theta_n = \text{id} - (\rho_n + \psi_n)$, $n \in \mathbb{N}$.

We show that (3) holds if the width of \mathcal{B} is at most l . Write $\mathcal{B} = \mathcal{B}_0 \cap \mathcal{A}$, where \mathcal{B}_0 is a masa-bimodule of width at most $l - 1$, while \mathcal{A} is a nest algebra bimodule. Letting

$$T \in \cap_{k \in \mathbb{N}} \overline{(\mathcal{B} \cap \mathcal{Z} \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}},$$

we see that $\tilde{\theta}_n(T) \in \mathcal{W}$ for each n , while

$$\tilde{\psi}_n(T) \in \cap_{k \in \mathbb{N}} \overline{(\mathcal{B}_0 \cap \mathcal{Z} \cap \mathcal{Z}_n \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}};$$

by the inductive assumption,

$$\tilde{\psi}_n(T) \in \overline{(\mathcal{B}_0 \cap \mathcal{Z}_n \cap \mathcal{Z} \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \overline{(\mathcal{B} \cap \mathcal{Z} \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

Letting S be a weak* cluster point of $(\tilde{\rho}_n(T))_{n \in \mathbb{N}}$, we have that

$$S \in \cap_{k \in \mathbb{N}} \overline{(\mathcal{B}_0 \cap \mathcal{Z} \cap \mathcal{N}_k \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}}$$

and hence, by the inductive assumption,

$$S \in \overline{(\mathcal{B} \cap \mathcal{Z} \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

We finally show that (4) holds if \mathcal{B} has width at most $l + 1$; this will complete the induction argument. Write $\mathcal{B} = \mathcal{B}_0 \cap \mathcal{A}$, where \mathcal{B}_0 is a masa-bimodule of width at most l and \mathcal{A} is a nest algebra bimodule. Letting

$$T \in \cap_{k \in \mathbb{N}} \overline{(\mathcal{B} \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}},$$

we see that $\tilde{\theta}_n(T) \in \mathcal{W}$ for each n , while

$$\tilde{\psi}_n(T) \in \cap_{k \in \mathbb{N}} \overline{(\mathcal{B}_0 \cap \mathcal{Z}_n \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

By the previous paragraph,

$$\tilde{\psi}_n(T) \in \overline{(\mathcal{B}_0 \cap \mathcal{Z}_n \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \overline{(\mathcal{B} \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

Let S be a weak* cluster point of $(\tilde{\rho}_n(T))_{n \in \mathbb{N}}$; then

$$S \in \cap_{k \in \mathbb{N}} \overline{(\mathcal{B}_0 \cap \mathcal{N}_k \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}}$$

and hence, by the inductive assumption, $S \in \overline{(\mathcal{B} \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}}$. \square

Theorem 3.7. *Let $\mathcal{V} \subseteq \mathcal{B}(H)$ be a weak* closed masa-bimodule, $\mathcal{B} \subseteq \mathcal{B}(H)$ be a masa-bimodule of finite width and $\mathcal{U} \subseteq \mathcal{B}(K)$ be a weak* closed subspace. Then*

$$\mathcal{F}(\overline{\mathcal{V} + \mathcal{B}}, \mathcal{U}) = \overline{\mathcal{F}(\mathcal{V}, \mathcal{U}) + \mathcal{B} \bar{\otimes} \mathcal{U}}.$$

In particular, if \mathcal{V} possesses property S_σ then so does $\overline{\mathcal{V} + \mathcal{B}}$.

Proof. Write $\mathcal{W} = \mathcal{F}(\mathcal{V}, \mathcal{U})$. We use induction on the width of \mathcal{B} . If the width of \mathcal{B} is zero, that is, if $\mathcal{B} = \mathcal{B}(H)$, then the identity follows from the fact that the space $\mathcal{B}(H)$ has property S_σ . Suppose that it holds for a masa-bimodule \mathcal{B} of finite width, and let \mathcal{A} be a nest algebra bimodule. Fix a decomposition scheme $(\rho_n, \psi_n, \mathcal{N}_n, \mathcal{Z}_n)_{n \in \mathbb{N}}$ for \mathcal{A} and set $\theta_n = \text{id} - (\rho_n + \psi_n)$, $n \in \mathbb{N}$. Let $T \in \mathcal{F}(\overline{\mathcal{V} + (\mathcal{B} \cap \mathcal{A})}, \mathcal{U})$. By Lemma 3.3, $\tilde{\theta}_n(T) \in \mathcal{W}$ while, using the monotonicity of the Fubini product and the inductive assumption, we have that

$$T \in \mathcal{F}(\overline{\mathcal{V} + \mathcal{B}}, \mathcal{U}) = \overline{\mathcal{B} \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

It follows by Lemma 3.2 that

$$\tilde{\psi}_n(T) \in \overline{(\mathcal{B} \cap \mathcal{Z}_n) \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \overline{(\mathcal{B} \cap \mathcal{A}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

Letting S be a weak* cluster point of $(\tilde{\rho}_n(T))_{n \in \mathbb{N}}$ and using Lemmas 3.2 and 3.6 (ii), we have

$$S \in \cap_{n \in \mathbb{N}} \overline{(\mathcal{B} \cap \mathcal{N}_n) \bar{\otimes} \mathcal{U} + \mathcal{W}} = \overline{(\mathcal{B} \cap (\cap_{n \in \mathbb{N}} \mathcal{N}_n)) \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \overline{(\mathcal{B} \cap \mathcal{A}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

We thus showed that

$$\mathcal{F}(\overline{\mathcal{V} + (\mathcal{B} \cap \mathcal{A})}, \mathcal{U}) \subseteq \overline{\mathcal{F}(\mathcal{V}, \mathcal{U}) + (\mathcal{B} \cap \mathcal{A}) \bar{\otimes} \mathcal{U}};$$

the converse inclusion is trivial.

If \mathcal{V} has S_σ then $\mathcal{F}(\mathcal{V}, \mathcal{U}) = \mathcal{V} \bar{\otimes} \mathcal{U}$ and hence, by the established identity,

$$\mathcal{F}(\overline{\mathcal{V} + \mathcal{B}}, \mathcal{U}) = \overline{\mathcal{V} \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{U}} = \overline{\mathcal{V} + \mathcal{B}} \bar{\otimes} \mathcal{U};$$

thus, $\overline{\mathcal{V} + \mathcal{B}}$ has property S_σ . \square

Remark. We note that, by [8], $\overline{\mathcal{V} + \mathcal{B}}$ is a reflexive masa-bimodule whenever \mathcal{V} is reflexive.

The next corollary is an immediate consequence of Theorem 3.7. It extends the fact, established in [14], that CSL algebras of finite width possess property S_σ .

Corollary 3.8. *If \mathcal{B}_i , $i = 1, \dots, n$, are masa-bimodules of finite width, then $\mathcal{B}_1 + \dots + \mathcal{B}_n$ has property S_σ .*

4. INTERSECTIONS AND SPANS

In this section, we establish an intersection formula for weak* closures of spans of subspaces of the form $\mathcal{B} \bar{\otimes} \mathcal{U}$, where \mathcal{B} is a masa-bimodule of finite width (see Theorem 4.4 and Corollary 4.21). This result will be used in Section 5 to study questions about operator synthesis.

Before formulating the main result of this section, Theorem 4.4, we state three propositions which will be needed in its proof. We first recall that the ranges of contractive Schur idempotents on $\mathcal{B}(H)$ are ternary masa-bimodules of the form $\oplus_k \mathcal{B}(E_k \mathcal{H}, F_k \mathcal{H})$, where $(E_k)_k \subseteq \mathcal{D}$ and $(F_k)_k \subseteq \mathcal{D}$ are families of mutually orthogonal projections (see, e.g., [15]).

Notation. For the rest of this section, we let $\mathcal{B}_i \subseteq \mathcal{B}(H)$ be a masa-bimodule of finite width, $\mathcal{U}, \mathcal{V}, \mathcal{U}_i$ be weak* closed subspaces of $\mathcal{B}(K)$, $i = 1, \dots, r$, and $\mathcal{W} = \sum_{i=1}^r \mathcal{B}_i \bar{\otimes} \mathcal{U}_i$.

Proposition 4.1. *Let $(\psi_i)_{i \in \mathbb{N}}$ be a nested sequence of contractive Schur idempotents.*

(i) *Let $(\phi_i)_{i \in \mathbb{N}}$ be a nested sequence of contractive Schur idempotents. Then the subspaces*

$$\cap_{k,i} \overline{(\text{Ran } \psi_i \cap \text{Ran } \phi_k) \bar{\otimes} \mathcal{U} + \text{Ran } \phi_k \bar{\otimes} \mathcal{V} + \mathcal{W}} \text{ and } \overline{((\cap_i \text{Ran } \psi_i) \cap (\cap_k \text{Ran } \phi_k)) \bar{\otimes} \mathcal{U} + (\cap_k \text{Ran } \phi_k) \bar{\otimes} \mathcal{V} + \mathcal{W}}$$

coincide.

(ii) *Let \mathcal{B} be a nest algebra bimodule. Then*

$$\cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V} + \mathcal{W}} = \overline{(\cap_i \text{Ran } \psi_i) \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

(iii) Let \mathcal{B} be a nest algebra bimodule. Then the subspaces

$$\overline{\cap_i \text{Ran } \psi_i \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_i \cap \mathcal{B}) \bar{\otimes} \mathcal{V} + \mathcal{W}} \quad \text{and} \\ \overline{(\cap_i \text{Ran } \psi_i) \bar{\otimes} \mathcal{U} + ((\cap_i \text{Ran } \psi_i) \cap \mathcal{B}) \bar{\otimes} \mathcal{V} + \mathcal{W}}$$

coincide.

(iv) Let \mathcal{B} be a masa-bimodule of finite width. Then the subspaces

$$\overline{\cap_i \text{Ran } \psi_i \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_i \cap \mathcal{B}) \bar{\otimes} \mathcal{V} + \mathcal{W}} \quad \text{and} \\ \overline{(\cap_i \text{Ran } \psi_i) \bar{\otimes} \mathcal{U} + ((\cap_i \text{Ran } \psi_i) \cap \mathcal{B}) \bar{\otimes} \mathcal{V} + \mathcal{W}}$$

coincide.

(v) $\overline{\cap_i \text{Ran } \psi_i \bar{\otimes} \mathcal{U} + \mathcal{W}} = \overline{(\cap_i \text{Ran } \psi_i) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$

We note that part (ii) of the previous proposition is more general than (v); however, for the purpose of its proof it will be convenient to formulate these statements separately.

Proposition 4.2. *Let \mathcal{M} be a ternary masa-bimodule and \mathcal{C} be a nest algebra bimodule. Then*

$$\overline{\mathcal{M} \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{\mathcal{C} \bar{\otimes} \mathcal{U} + \mathcal{W}} = \overline{(\mathcal{M} \cap \mathcal{C}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

Proposition 4.3. *Let \mathcal{M} be a ternary masa-bimodule and \mathcal{C} be a masa-bimodule of finite width. Then*

$$\overline{\mathcal{M} \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{\mathcal{C} \bar{\otimes} \mathcal{U} + \mathcal{W}} = \overline{(\mathcal{M} \cap \mathcal{C}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

Theorem 4.4. *Let $\mathcal{C}_j \subseteq \mathcal{B}(H)$ be a masa-bimodule of finite width, $j = 1, \dots, m$. Then*

$$\overline{\cap_{j=1}^m \mathcal{C}_j \bar{\otimes} \mathcal{U} + \mathcal{W}} = \overline{(\cap_{j=1}^m \mathcal{C}_j) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

The proof of the above results will be given simultaneously, using induction on the number r of terms in the sum $\mathcal{W} = \sum_{i=1}^r \mathcal{B}_i \bar{\otimes} \mathcal{U}_i$ and will be split into a number of lemmas. The first series of steps, namely Lemmas 4.5–4.12, provide the base of the induction. We will refer to the statements in Proposition 4.1 by their corresponding numbers (i) – (v). It will be convenient to assume that $\mathcal{W} = \{0\}$ when $r = 0$.

Given the notation in Proposition 4.1, throughout the proofs, we will set for brevity

$$\mathcal{N} = \cap_i \text{Ran } \psi_i \quad \text{and} \quad \mathcal{R} = \cap_k \text{Ran } \phi_k.$$

Lemma 4.5. *Proposition 4.1 (i) holds if $r = 0$.*

Proof. Let

$$\Omega = \overline{(\mathcal{N} \cap \mathcal{R}) \bar{\otimes} \mathcal{U} + \mathcal{R} \bar{\otimes} \mathcal{V}}$$

and fix

$$X \in \cap_{k,i} \overline{(\text{Ran } \psi_i \cap \text{Ran } \phi_k) \bar{\otimes} \mathcal{U} + \text{Ran } \phi_k \bar{\otimes} \mathcal{V}}.$$

By Lemma 3.1, $\tilde{\psi}_i^\perp(X) \in \text{Ran } \phi_k \bar{\otimes} \mathcal{V}$ for all k, i . By Lemma 3.5, $\tilde{\psi}_i^\perp(X) \in \mathcal{R} \bar{\otimes} \mathcal{V} \subseteq \Omega$, $i \in \mathbb{N}$. On the other hand, for all $i \in \mathbb{N}$, we have, by Lemma 3.1,

$$\begin{aligned} \tilde{\psi}_i(X) &\in \overline{(\text{Ran } \psi_i \cap \text{Ran } \phi_i) \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_i \cap \text{Ran } \phi_i) \bar{\otimes} \mathcal{V}} \\ &= (\text{Ran } \psi_i \cap \text{Ran } \phi_i) \bar{\otimes} (\overline{\mathcal{U} + \mathcal{V}}). \end{aligned}$$

By Lemma 3.5, any weak* cluster point S of the sequence $(\tilde{\psi}_i(X))_{i \in \mathbb{N}}$ belongs to $(\mathcal{N} \cap \mathcal{R}) \bar{\otimes} (\overline{\mathcal{U} + \mathcal{V}})$, a subset of Ω . Thus, $X = (X - S) + S \in \Omega$. \square

Lemma 4.6. *Proposition 4.1 (ii) holds if $r = 0$.*

Proof. Let $\Omega = \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}$ and fix

$$X \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}.$$

Let $(\phi_k, \theta_k, \mathcal{M}_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{B} . By Lemma 3.1,

$$\tilde{\phi}_k(X) \in \cap_i \overline{(\text{Ran } \psi_i \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{M}_k \bar{\otimes} \mathcal{V}}.$$

If S is a weak* cluster point of the sequence $(\tilde{\phi}_k(X))_{k \in \mathbb{N}}$, then

$$S \in \cap_{k,i} \overline{(\text{Ran } \psi_i \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{M}_k \bar{\otimes} \mathcal{V}}.$$

By Lemma 4.5, $S \in \Omega$. It hence suffices to prove that $\tilde{\phi}_k^\perp(X) \in \Omega$ for all $k \in \mathbb{N}$. Observe that

$$\tilde{\phi}_k^\perp(X) \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + \mathcal{Z}_k \bar{\otimes} \mathcal{V}}.$$

Using Lemmas 3.1 and 3.5, we see that

$$\tilde{\theta}_k^\perp(\tilde{\phi}_k^\perp(X)) \in \cap_i (\text{Ran } \psi_i \bar{\otimes} \mathcal{U}) = \mathcal{N} \bar{\otimes} \mathcal{U} \subseteq \Omega.$$

Write $\theta_k = \sum_{p=1}^k \theta_{k,p}$, where $\theta_{k,p}$ is a contractive Schur idempotent whose range is contained in \mathcal{B} (see the last paragraph of Section 2). Then

$$\tilde{\theta}_{k,p}(\tilde{\phi}_k^\perp(X)) \in \cap_i \overline{(\text{Ran } \psi_i \cap \text{Ran } \theta_{k,p}) \bar{\otimes} \mathcal{U} + \text{Ran } \theta_{k,p} \bar{\otimes} \mathcal{V}}.$$

By Lemma 4.5,

$$\tilde{\theta}_{k,p}(\tilde{\phi}_k^\perp(X)) \in \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + \text{Ran } \theta_{k,p} \bar{\otimes} \mathcal{V}} \subseteq \Omega.$$

Hence, $\tilde{\phi}_k^\perp(X) = \tilde{\theta}_k^\perp(\tilde{\phi}_k^\perp(X)) + \sum_{p=1}^k \tilde{\theta}_{k,p}(\tilde{\phi}_k^\perp(X)) \in \Omega$ for all $k \in \mathbb{N}$ and the proof is complete. \square

Lemma 4.7. *Proposition 4.1 (iii) holds if $r = 0$.*

Proof. Let $\Omega = \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap \mathcal{B}) \bar{\otimes} \mathcal{V}}$ and fix

$$X \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_i \cap \mathcal{B}) \bar{\otimes} \mathcal{V}}.$$

Let $(\phi_k, \theta_k, \mathcal{M}_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{B} . By Lemma 3.1,

$$\tilde{\phi}_k^\perp(X) \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_i \cap \mathcal{Z}_k) \bar{\otimes} \mathcal{V}}.$$

By Lemmas 3.1 and 3.5,

$$\tilde{\theta}_k^\perp(\tilde{\phi}_k^\perp(X)) \in \cap_i (\text{Ran } \psi_i \bar{\otimes} \mathcal{U}) = \mathcal{N} \bar{\otimes} \mathcal{U} \subseteq \Omega,$$

and by Lemmas 3.1, 3.2 and 3.5,

$$\begin{aligned}\tilde{\theta}_k(\tilde{\phi}_k^\perp(X)) &\in \cap_i ((\text{Ran } \psi_i \cap \mathcal{Z}_k) \bar{\otimes} \overline{\mathcal{U} + \mathcal{V}}) \\ &\subseteq (\cap_i (\text{Ran } \psi_i \bar{\otimes} \overline{\mathcal{U} + \mathcal{V}})) \cap (\mathcal{Z}_k \bar{\otimes} \overline{\mathcal{U} + \mathcal{V}}) \\ &= (\mathcal{N} \bar{\otimes} \overline{\mathcal{U} + \mathcal{V}}) \cap (\mathcal{Z}_k \bar{\otimes} \overline{\mathcal{U} + \mathcal{V}}) = (\mathcal{N} \cap \mathcal{Z}_k) \bar{\otimes} \overline{\mathcal{U} + \mathcal{V}} \subseteq \Omega;\end{aligned}$$

thus, $\tilde{\phi}_k^\perp(X) \in \Omega$. On the other hand, by Lemma 3.1,

$$\tilde{\phi}_k(X) \in (\text{Ran } \psi_k \cap \text{Ran } \phi_k) \bar{\otimes} \overline{\mathcal{U} + \mathcal{V}}, \quad k \in \mathbb{N}.$$

Therefore, if S is a weak* cluster point of the sequence $(\tilde{\phi}_k(X))_{k \in \mathbb{N}}$, then, by Lemma 3.5,

$$S \in (\mathcal{N} \cap \mathcal{R}) \bar{\otimes} \overline{\mathcal{U} + \mathcal{V}} \subseteq \Omega.$$

The proof is complete. \square

Lemma 4.8. *Proposition 4.1 (iv) holds if $r = 0$.*

Proof. Let

$$\Omega = \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap \mathcal{B}) \bar{\otimes} \mathcal{V}}.$$

We use induction on the width n of \mathcal{B} . If $n = 1$, the conclusion follows from Lemma 4.7. Suppose that the statement holds for masa-bimodules of width at most $n - 1$ and let $\mathcal{B} = \cap_{l=1}^n \mathcal{A}_l$, where \mathcal{A}_l is a nest algebra bimodule, $l = 1, \dots, n$. Fix

$$X \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_i \cap \mathcal{B}) \bar{\otimes} \mathcal{V}}.$$

By the inductive assumption, X belongs to both

$$\overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap (\cap_{l=1}^{n-1} \mathcal{A}_l)) \bar{\otimes} \mathcal{V}}$$

and

$$\overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap \mathcal{A}_n) \bar{\otimes} \mathcal{V}}.$$

Let $(\phi_k, \theta_k, \mathcal{M}_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{A}_n . For a fixed k , we have that $\tilde{\phi}_k^\perp(X)$ belongs to the intersection of the spaces

$$\overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap (\cap_{l=1}^{n-1} \mathcal{A}_l)) \bar{\otimes} \mathcal{V}} \quad \text{and} \quad \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap \mathcal{Z}_k) \bar{\otimes} \mathcal{V}}.$$

By Lemma 3.1,

$$\tilde{\theta}_k^\perp(\tilde{\phi}_k^\perp(X)) \in \mathcal{N} \bar{\otimes} \mathcal{U} \subseteq \Omega$$

and

$$\tilde{\theta}_k(\tilde{\phi}_k^\perp(X)) \in \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap (\cap_{i=1}^{n-1} \mathcal{A}_i) \cap \mathcal{Z}_k) \bar{\otimes} \mathcal{U}} \subseteq \Omega,$$

since $\mathcal{Z}_k \subseteq \mathcal{A}_n$. Thus,

$$\tilde{\phi}_k^\perp(X) = \tilde{\theta}_k(\tilde{\phi}_k^\perp(X)) + \tilde{\theta}_k^\perp(\tilde{\phi}_k^\perp(X)) \in \Omega.$$

Let S be a weak* cluster point of $(\tilde{\phi}_k(X))_{k \in \mathbb{N}}$. Then

$$S \in \cap_k \overline{(\text{Ran } \psi_k \cap \text{Ran } \phi_k) \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_k \cap \text{Ran } \phi_k \cap (\cap_{l=1}^{n-1} \mathcal{A}_l)) \bar{\otimes} \mathcal{V}}.$$

By the inductive assumption, the latter space coincides with

$$\overline{(\mathcal{N} \cap \mathcal{R}) \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap \mathcal{R}) \cap (\cap_{l=1}^{n-1} \mathcal{A}_l) \bar{\otimes} \mathcal{V}},$$

which is a subset of Ω since $\mathcal{R} \subseteq \mathcal{A}_n$. \square

Lemma 4.9. *Proposition 4.1 (v) holds if $r = 1$.*

Proof. Let $\Omega = \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + \mathcal{W}}$. We set $\mathcal{B} = \mathcal{B}_1$ and $\mathcal{V} = \mathcal{U}_1$, and use induction on the width n of \mathcal{B} . For $n = 1$ the conclusion follows from Lemma 4.6. Suppose that the statement holds if the length of \mathcal{B} does not exceed $n-1$ and assume that $\mathcal{B} = \cap_{l=1}^n \mathcal{A}_l$ where \mathcal{A}_l is a nest algebra bimodule, $l = 1, \dots, n$. Let $(\phi_k, \theta_k, \mathcal{M}_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{A}_n . Fix

$$X \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}.$$

By the inductive assumption, X belongs to the intersection of the spaces

$$\overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\cap_{l=1}^{n-1} \mathcal{A}_l) \bar{\otimes} \mathcal{V}} \quad \text{and} \quad \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + \mathcal{A}_n \bar{\otimes} \mathcal{V}}.$$

By Lemma 3.1, for a fixed k , $\tilde{\phi}_k^\perp(X) \in \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + \mathcal{Z}_k \bar{\otimes} \mathcal{V}}$. Therefore, $\tilde{\theta}_k^\perp(\tilde{\phi}_k^\perp(X)) \in \mathcal{N} \bar{\otimes} \mathcal{U}$, and hence $\tilde{\theta}_k^\perp(\tilde{\phi}_k^\perp(X)) \in \Omega$. On the other hand,

$$\tilde{\theta}_k(\tilde{\phi}_k^\perp(X)) \in \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + ((\cap_{l=1}^{n-1} \mathcal{A}_l) \cap \mathcal{Z}_k) \bar{\otimes} \mathcal{V}} \subseteq \Omega$$

since $\mathcal{Z}_k \subseteq \mathcal{A}_n$. Thus, $\tilde{\phi}_k^\perp(X) \in \Omega$, for all $k \in \mathbb{N}$. It hence suffices to prove that, if S is the limit of a subsequence $(\tilde{\phi}_{k_l}(X))_{l \in \mathbb{N}}$, then $S \in \Omega$. Let S' be a weak* cluster point of the sequence $(\tilde{\psi}_{k_l}(\tilde{\phi}_{k_l}(X)))_{l \in \mathbb{N}}$; then $S'' = S - S'$ is a weak* cluster point of $(\tilde{\psi}_{k_l}^\perp(\tilde{\phi}_{k_l}(X)))_{l \in \mathbb{N}}$. By Lemma 3.1,

$$\tilde{\psi}_k(\tilde{\phi}_k(X)) \in \overline{(\mathcal{M}_k \cap \text{Ran } \psi_k) \bar{\otimes} \mathcal{U} + ((\cap_{l=1}^{n-1} \mathcal{A}_l) \cap (\mathcal{M}_k \cap \text{Ran } \psi_k)) \bar{\otimes} \mathcal{V}},$$

while

$$\tilde{\psi}_k^\perp(\tilde{\phi}_k(X)) = \tilde{\phi}_k(\tilde{\psi}_k^\perp(X)) \in \mathcal{B} \bar{\otimes} \mathcal{V} \subseteq \Omega, \quad k \in \mathbb{N}.$$

On the other hand, Lemma 4.8 implies that

$$S' \in \overline{((\cap_k \mathcal{M}_k) \cap \mathcal{N}) \bar{\otimes} \mathcal{U} + ((\cap_{l=1}^{n-1} \mathcal{A}_l) \cap (\cap_k \mathcal{M}_k) \cap \mathcal{N}) \bar{\otimes} \mathcal{V}} \subseteq \Omega.$$

Thus, $S = S' + S'' \in \Omega$. □

Lemma 4.10. *Proposition 4.2 holds if $r = 1$.*

Proof. Set $\mathcal{B} = \mathcal{B}_1$ and $\mathcal{V} = \mathcal{U}_1$, let $\Omega = \overline{(\mathcal{M} \cap \mathcal{C}) \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}$ and fix

$$X \in \overline{\mathcal{M} \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V} \cap \overline{\mathcal{C} \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}}.$$

Let $(\psi_k, \theta_k, \text{Ran } \psi_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{C} . We have that

$$\tilde{\psi}_k^\perp(X) \in \overline{\mathcal{M} \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V} \cap \overline{\mathcal{Z}_k \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}}.$$

By Lemma 3.2,

$$\tilde{\psi}_k^\perp(X) \in \overline{(\mathcal{M} \cap \mathcal{Z}_k) \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}} \subseteq \overline{(\mathcal{M} \cap \mathcal{C}) \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}.$$

On the other hand, by Lemmas 3.1 and 3.2,

$$\tilde{\psi}_k(X) \in \overline{(\mathcal{M}_k \cap \text{Ran } \psi_k) \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}$$

for all $k \in \mathbb{N}$. Lemma 4.9 shows that, if S is a weak* cluster point of the sequence $(\tilde{\psi}_k(X))_{k \in \mathbb{N}}$, then

$$S \in \overline{(\cap_k (\mathcal{M}_k \cap \text{Ran } \psi_k)) \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}.$$

Since $\cap_k(\mathcal{M}_k \cap \text{Ran } \psi_k) \subseteq \mathcal{M} \cap \mathcal{C}$, we conclude that $S \in \Omega$. The proof is complete. \square

Lemma 4.11. *Proposition 4.3 holds if $r = 1$.*

Proof. Write $\mathcal{B} = \mathcal{B}_1$ and $\mathcal{V} = \mathcal{U}_1$. We use induction on the width n of \mathcal{C} . If $n = 1$, the statement reduces to Lemma 4.10. Suppose that the statement holds for all masa-bimodules \mathcal{C} of width at most $n - 1$ and let $\mathcal{C} = \cap_{l=1}^n \mathcal{C}_l$, where \mathcal{C}_l is a nest algebra bimodule, $l = 1, \dots, n$. Fix

$$X \in \overline{\mathcal{M} \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}} \cap \overline{\mathcal{C} \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}}.$$

Let $(\psi_k, \theta_k, \text{Ran } \psi_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{C}_n and recall that $\mathcal{N} = \cap_k \text{Ran } \psi_k$. There exists a descending sequence $(\mathcal{M}_k)_{k \in \mathbb{N}}$ of ranges of contractive Schur idempotents such that $\mathcal{M} = \cap_{k \in \mathbb{N}} \mathcal{M}_k$. Observe that, by the inductive assumption,

$$\begin{aligned} \tilde{\psi}_k^\perp(X) &\in \overline{\mathcal{M} \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}} \cap \overline{(\cap_{l=1}^{n-1} \mathcal{C}_l) \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}} \cap \overline{\mathcal{Z}_k \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}} \\ &= \overline{(\mathcal{M} \cap (\cap_{l=1}^{n-1} \mathcal{C}_l) \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}) \cap (\mathcal{Z}_k \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V})}. \end{aligned}$$

By Lemma 3.2,

$$\tilde{\psi}_k^\perp(X) \in \overline{(\mathcal{M} \cap \mathcal{C}) \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}}.$$

Observe that

$$\tilde{\psi}_k(X) \in \overline{\mathcal{M}_k \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}} \cap \overline{(\cap_{l=1}^{n-1} \mathcal{C}_l) \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}} \cap \overline{\text{Ran } \psi_k \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}}$$

and so, by Lemma 3.2,

$$\tilde{\psi}_k(X) \in \overline{(\mathcal{M}_k \cap \text{Ran } \psi_k) \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}} \cap \overline{(\cap_{l=1}^{n-1} \mathcal{C}_l) \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}}, \quad k \in \mathbb{N}.$$

By Lemma 4.9, if S is a weak* cluster point of $(\tilde{\psi}_k(X))_{k \in \mathbb{N}}$, then

$$S \in \overline{(\mathcal{M} \cap \mathcal{N}) \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}} \cap \overline{(\cap_{l=1}^{n-1} \mathcal{C}_l) \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}}.$$

By the inductive assumption,

$$S \in \overline{(\mathcal{M} \cap \mathcal{N} \cap (\cap_{l=1}^{n-1} \mathcal{C}_l)) \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}} \subseteq \overline{(\mathcal{M} \cap \mathcal{C}) \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}},$$

since $\mathcal{N} \subseteq \mathcal{C}_n$. The proof is complete. \square

Lemma 4.12. *Theorem 4.4 holds if $r = 1$.*

Proof. Since each masa-bimodule of finite width is the finite intersection of nest algebra masa-bimodules, we may assume, without loss of generality, that \mathcal{C}_j is a nest algebra bimodule, $j = 1, \dots, m$. Set $\mathcal{B} = \mathcal{B}_1$, $\mathcal{V} = \mathcal{U}_1$ and $\mathcal{C} = \cap_{j=1}^m \mathcal{C}_j$. We use induction on m . For $m = 1$, the statement is trivial; suppose it holds if the number of given bimodules is at most $m - 1$ and fix $X \in \cap_{j=1}^m \overline{\mathcal{C}_j \otimes \mathcal{U} + \mathcal{B} \otimes \mathcal{V}}$. Let $(\phi_k, \theta_k, \mathcal{M}_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{C}_m and set $\mathcal{M} = \cap_{k \in \mathbb{N}} \mathcal{M}_k$. Since $X \in \overline{\mathcal{C}_m \otimes \mathcal{U} + \mathcal{W}}$, Lemma 3.1 implies that

$$\tilde{\phi}_k^\perp(X) \in \overline{\mathcal{Z}_k \otimes \mathcal{U} + \mathcal{W}}.$$

By the inductive assumption and the invariance of $\cap_{j=1}^{m-1} \mathcal{C}_j$ under ϕ_k , we have that

$$\tilde{\phi}_k^\perp(X) \in \overline{(\cap_{j=1}^{m-1} \mathcal{C}_j) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

Hence, by Lemma 3.2,

$$\tilde{\phi}_k^\perp(X) \in \overline{((\cap_{i=1}^{m-1} \mathcal{C}_i) \cap \mathcal{Z}_k) \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}} \subseteq \overline{\mathcal{C} \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}, \quad k \in \mathbb{N}.$$

On the other hand,

$$\tilde{\phi}_k(X) \in \overline{\mathcal{M}_k \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}} \cap \overline{(\cap_{j=1}^{m-1} \mathcal{C}_j) \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}, \quad k \in \mathbb{N}.$$

If S is a weak* cluster point of the sequence $(\tilde{\phi}_k(X))_{k \in \mathbb{N}}$, by Lemma 4.9 we have

$$S \in \overline{\mathcal{M} \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}} \cap \overline{(\cap_{j=1}^{m-1} \mathcal{C}_j) \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}.$$

By Lemma 4.11,

$$S \in \overline{(\mathcal{M} \cap (\cap_{i=1}^{m-1} \mathcal{C}_i)) \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}} \subseteq \overline{\mathcal{C} \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}}.$$

□

We next establish the induction step for the proofs of Propositions 4.1 – 4.3 and Theorem 4.4; this is done in Lemmas 4.13 – 4.20 below. To this end, we assume that the statements in Proposition 4.1 (i)–(iv) hold if the space \mathcal{W} has $r - 1$ summands, while Proposition 4.1 (v), Proposition 4.2, Proposition 4.3 and Theorem 4.4 hold if \mathcal{W} has r summands.

Lemma 4.13. *Proposition 4.1 (i) holds if the space \mathcal{W} has r terms.*

Proof. Let

$$\Omega = \overline{(\mathcal{N} \cap \mathcal{R}) \bar{\otimes} \mathcal{U} + \mathcal{R} \bar{\otimes} \mathcal{V} + \mathcal{W}}$$

and fix

$$X \in \cap_{k,i} \overline{(\text{Ran } \psi_i \cap \text{Ran } \phi_k) \bar{\otimes} \mathcal{U} + \text{Ran } \phi_k \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

By Lemma 3.1, $\tilde{\psi}_i^\perp(X) \in \overline{\text{Ran } \phi_k \bar{\otimes} \mathcal{V} + \mathcal{W}}$ for all $k, i \in \mathbb{N}$. By the inductive assumption concerning Proposition 4.1 (v),

$$\tilde{\psi}_i^\perp(X) \in \overline{\mathcal{R} \bar{\otimes} \mathcal{V} + \mathcal{W}} \subseteq \Omega, \quad i \in \mathbb{N}.$$

On the other hand, for all $k \in \mathbb{N}$ we have

$$\begin{aligned} \tilde{\psi}_k(X) &\in \overline{(\text{Ran } \psi_k \cap \text{Ran } \phi_k) \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_k \cap \text{Ran } \phi_k) \bar{\otimes} \mathcal{V} + \mathcal{W}} \\ &= \overline{(\text{Ran } \psi_k \cap \text{Ran } \phi_k) \bar{\otimes} (\mathcal{U} + \mathcal{V}) + \mathcal{W}}. \end{aligned}$$

Let S be a weak* cluster point of $(\tilde{\psi}_k(X))_{k \in \mathbb{N}}$. Once again by the inductive assumption concerning Proposition 4.1 (v),

$$S \in \overline{(\mathcal{N} \cap \mathcal{R}) \bar{\otimes} (\mathcal{U} + \mathcal{V}) + \mathcal{W}}.$$

Thus, $S \in \Omega$ and the proof is complete. □

Lemma 4.14. *Proposition 4.1 (ii) holds if the space \mathcal{W} has r terms.*

Proof. Let

$$\Omega = \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V} + \mathcal{W}}$$

and fix

$$X \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

Let $(\phi_k, \theta_k, \text{Ran } \phi_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{B} and observe that, by Lemma 3.1, we have

$$\tilde{\phi}_k(X) \in \cap_i \overline{(\text{Ran } \psi_i \cap \text{Ran } \phi_k) \bar{\otimes} \mathcal{U} + \text{Ran } \phi_k \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

Letting S be a weak* cluster point of the sequence $(\tilde{\phi}_k(X))_{k \in \mathbb{N}}$, we have that

$$S \in \cap_{k,i} \overline{(\text{Ran } \psi_i \cap \text{Ran } \phi_k) \bar{\otimes} \mathcal{U} + \text{Ran } \phi_k \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

By Lemma 4.13 and the fact that $\mathcal{R} = \cap_k \text{Ran } \phi_k \subseteq \mathcal{B}$, we have that $S \in \Omega$. So it suffices to prove that $\tilde{\phi}_k^\perp(X) \in \Omega$ for all $k \in \mathbb{N}$. Note that, by Lemma 3.1,

$$\tilde{\phi}_k^\perp(X) \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + \mathcal{Z}_k \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

By the inductive assumption concerning Proposition 4.1 (v) and Lemma 3.1 again,

$$\tilde{\theta}_k^\perp(\tilde{\phi}_k^\perp(X)) \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + \mathcal{W}} = \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \Omega.$$

Write $\theta_k = \sum_{p=1}^k \theta_{k,p}$, where each $\theta_{k,p}$ is a contractive Schur idempotent whose range is contained in \mathcal{B} . We have that

$$\tilde{\theta}_{k,p}(\tilde{\phi}_k^\perp(X)) \in \cap_i \overline{(\text{Ran } \psi_i \cap \text{Ran } \theta_{k,p}) \bar{\otimes} \mathcal{U} + (\text{Ran } \theta_{k,p}) \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

By Lemma 4.13 (applied in the case of a constant sequence of maps with term $\theta_{k,p}$), we have

$$\tilde{\theta}_{k,p}(\tilde{\phi}_k^\perp(X)) \in \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\text{Ran } \theta_{k,p}) \bar{\otimes} \mathcal{V} + \mathcal{W}} \subseteq \Omega, \quad p = 1, \dots, k.$$

It follows that $\tilde{\theta}_k(\tilde{\phi}_k^\perp(X)) \in \Omega$ and hence $\tilde{\phi}_k^\perp(X) = \tilde{\theta}_k(\tilde{\phi}_k^\perp(X)) + \tilde{\theta}_k^\perp(\tilde{\phi}_k^\perp(X)) \in \Omega$. \square

Lemma 4.15. *Proposition 4.1 (iii) holds if the space \mathcal{W} has r terms.*

Proof. We let

$$\Omega = \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap \mathcal{B}) \bar{\otimes} \mathcal{V} + \mathcal{W}}$$

and fix

$$X \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_i \cap \mathcal{B}) \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

Let $(\phi_k, \theta_k, \mathcal{M}_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{B} and observe that

$$\tilde{\phi}_k^\perp(X) \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_i \cap \mathcal{Z}_k) \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

Using the inductive assumption concerning Proposition 4.1 (v), we have

$$\tilde{\theta}_k^\perp(\tilde{\phi}_k^\perp(X)) \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + \mathcal{W}} = \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \Omega.$$

Write $\theta_k = \sum_{p=1}^k \theta_{k,p}$ as in the proof of Lemma 4.14; then

$$\begin{aligned} \tilde{\theta}_{k,p}(\tilde{\phi}_k^\perp(X)) &\in \cap_i \overline{(\text{Ran } \psi_i \cap \text{Ran } \theta_{k,p}) \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_i \cap \text{Ran } \theta_{k,p}) \bar{\otimes} \mathcal{V} + \mathcal{W}} \\ &= \cap_i \overline{(\text{Ran } \psi_i \cap \text{Ran } \theta_{k,p}) \bar{\otimes} (\mathcal{U} + \mathcal{V}) + \mathcal{W}}. \end{aligned}$$

By the inductive assumption concerning Proposition 4.1 (v), we have that $\tilde{\theta}_{k,p}(\tilde{\phi}_k^\perp(X)) \in \Omega$ for each $p = 1, \dots, k$. It follows that $\tilde{\phi}_k^\perp(X) \in \Omega$. Let S be a weak* cluster point of the sequence $(\tilde{\phi}_k(X))_{k \in \mathbb{N}}$. Since

$$\tilde{\phi}_k(X) \in \overline{\cap_k (\text{Ran } \psi_k \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + \mathcal{V} + \mathcal{W}},$$

by the same inductive assumption once again,

$$S \in \overline{(\mathcal{N} \cap (\cap_k \mathcal{M}_k)) \bar{\otimes} \mathcal{U} + \mathcal{V} + \mathcal{W}} \subseteq \Omega.$$

□

Lemma 4.16. *Proposition 4.1 (iv) holds if the space \mathcal{W} has r terms.*

Proof. Let

$$\Omega = \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap \mathcal{B}) \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

We use induction on the width n of \mathcal{B} . The case $n = 1$ reduces to Lemma 4.15. Suppose that the statement holds for masa-bimodules of width at most $n - 1$ and let $\mathcal{B} = \cap_{l=1}^n \mathcal{C}_l$ where every \mathcal{C}_l is nest algebra bimodule, $l = 1, \dots, n$. Fix

$$X \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_i \cap \mathcal{B}) \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

By the inductive assumption, X belongs to the intersection of

$$\overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap (\cap_{l=1}^{n-1} \mathcal{C}_l)) \bar{\otimes} \mathcal{V} + \mathcal{W}} \quad \text{and} \quad \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap \mathcal{C}_n) \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

Let $(\phi_k, \theta_k, \mathcal{M}_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{C}_n . For a fixed $k \in \mathbb{N}$, we have that $\tilde{\phi}_k^\perp(X)$ belongs to the intersection of

$$\overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap (\cap_{l=1}^{n-1} \mathcal{C}_l)) \bar{\otimes} \mathcal{V} + \mathcal{W}} \quad \text{and} \quad \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap \mathcal{Z}_k) \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

By Lemma 3.2,

$$\tilde{\phi}_k^\perp(X) \in \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap (\cap_{i=1}^{n-1} \mathcal{C}_i) \cap \mathcal{Z}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \Omega.$$

Let S be a weak* cluster point of the sequence $(\tilde{\phi}_k(X))_{k \in \mathbb{N}}$; we have

$$S \in \overline{\cap_k (\text{Ran } \psi_k \cap \mathcal{M}_k) \bar{\otimes} \mathcal{U} + (\text{Ran } \psi_k \cap \mathcal{M}_k \cap (\cap_{l=1}^{n-1} \mathcal{C}_l)) \bar{\otimes} \mathcal{V} + \mathcal{W}}.$$

By the inductive assumption, the latter space is equal to

$$\overline{(\mathcal{N} \cap (\cap_k \mathcal{M}_k)) \bar{\otimes} \mathcal{U} + (\mathcal{N} \cap (\cap_k \mathcal{M}_k) \cap (\cap_{l=1}^{n-1} \mathcal{C}_l)) \bar{\otimes} \mathcal{V} + \mathcal{W}}$$

which is contained in Ω since $\cap_k \mathcal{M}_k \subseteq \mathcal{C}_n$. □

Lemma 4.17. *Proposition 4.1 (v) holds if the space \mathcal{W} has $r + 1$ terms.*

Proof. Let $\Omega = \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + \mathcal{W}}$. Set $\mathcal{B} = \mathcal{B}_{r+1}$, $\mathcal{V} = \mathcal{U}_{r+1}$ and $\mathcal{W}_0 = \sum_{i=1}^r \mathcal{B}_i \bar{\otimes} \mathcal{U}_i$. We use induction on the width n of \mathcal{B} . For $n = 1$ the conclusion follows from the inductive assumption concerning Proposition 4.1 (ii). Assume that it holds when the length of \mathcal{B} does not exceed $n - 1$, and suppose that $\mathcal{B} = \cap_{l=1}^n \mathcal{C}_l$ where \mathcal{C}_l is a nest algebra bimodule, $l = 1, \dots, n$. Let $(\phi_k, \theta_k, \mathcal{M}_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{C}_n . Fix

$$X \in \cap_i \overline{\text{Ran } \psi_i \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V} + \mathcal{W}_0}.$$

By assumption,

$$X \in \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\cap_{l=1}^{n-1} \mathcal{C}_l) \bar{\otimes} \mathcal{V} + \mathcal{W}_0 \cap \mathcal{N} \bar{\otimes} \mathcal{U} + \mathcal{C}_n \bar{\otimes} \mathcal{V} + \mathcal{W}_0}.$$

For a fixed $k \in \mathbb{N}$, we have

$$\tilde{\phi}_k^\perp(X) \in \overline{\mathcal{N} \bar{\otimes} \mathcal{U} + (\cap_{l=1}^{n-1} \mathcal{C}_l) \bar{\otimes} \mathcal{V} + \mathcal{W}_0 \cap \mathcal{N} \bar{\otimes} \mathcal{U} + \mathcal{Z}_k \bar{\otimes} \mathcal{V} + \mathcal{W}_0}.$$

By Lemma 3.2, $\tilde{\phi}_k^\perp(X) \in \Omega$. Let S be a weak* cluster point of the sequence $(\tilde{\phi}_k(X))_{k \in \mathbb{N}}$, and S' and S'' be weak* cluster points of $(\tilde{\psi}_k^\perp(\tilde{\phi}_k(X)))_{k \in \mathbb{N}}$ and $(\tilde{\psi}_k(\tilde{\phi}_k(X)))_{k \in \mathbb{N}}$, respectively, such that $S = S' + S''$.

We have

$$\tilde{\phi}_k(X) \in \overline{(\mathcal{M}_k \cap \mathcal{N}) \bar{\otimes} \mathcal{U} + ((\cap_{l=1}^{n-1} \mathcal{C}_l) \cap \mathcal{M}_k) \bar{\otimes} \mathcal{V} + \mathcal{W}_0}.$$

It follows that

$$\tilde{\psi}_k^\perp(\tilde{\phi}_k(X)) \in \overline{((\cap_{l=1}^{n-1} \mathcal{C}_l) \cap \mathcal{M}_k) \bar{\otimes} \mathcal{V} + \mathcal{W}_0}, \quad k \in \mathbb{N},$$

and hence, by the inductive assumption concerning Proposition 4.1 (v),

$$S' \in \overline{(\cap_{l=1}^{n-1} \mathcal{C}_l) \bar{\otimes} \mathcal{V} + \mathcal{W}_0 \cap (\cap_k \mathcal{M}_k) \bar{\otimes} \mathcal{V} + \mathcal{W}_0}.$$

It follows from the inductive assumption concerning Proposition 4.3 that $S' \in \Omega$. On the other hand,

$$S'' \in \overline{\cap_k (\mathcal{M}_k \cap \text{Ran } \psi_k) \bar{\otimes} \mathcal{U} + ((\cap_{l=1}^{n-1} \mathcal{C}_l) \cap \mathcal{M}_k \cap \text{Ran } \psi_k) \bar{\otimes} \mathcal{V} + \mathcal{W}_0}.$$

Since $\|\phi_k \psi_k\| \leq 1$ for each $k \in \mathbb{N}$, Lemma 4.16 implies that $S'' \in \Omega$. It now follows that $S \in \Omega$. \square

Lemma 4.18. *Proposition 4.2 holds if the space \mathcal{W} has $r+1$ terms.*

Proof. Let $\Omega = \overline{(\mathcal{M} \cap \mathcal{C}) \bar{\otimes} \mathcal{U} + \mathcal{W}}$ and fix

$$X \in \overline{\mathcal{M} \bar{\otimes} \mathcal{U} + \mathcal{W} \cap \overline{\mathcal{C} \bar{\otimes} \mathcal{U} + \mathcal{W}}}.$$

Let $(\psi_k, \theta_k, \text{Ran } \psi_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{C} . Let $(\phi_k)_{k \in \mathbb{N}}$ be a nested sequence of contractive Schur idempotents such that $\mathcal{M} = \cap_{k=1}^\infty \mathcal{M}_k$, where $\mathcal{M}_k = \text{Ran } \phi_k$, $k \in \mathbb{N}$. We first observe that, by Lemma 3.2,

$$\tilde{\theta}_k(X) \in \overline{(\mathcal{Z}_k \cap \mathcal{M}) \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \Omega, \quad k \in \mathbb{N}.$$

On the other hand, by Lemma 3.1,

$$(\tilde{\psi}_k + \tilde{\theta}_k)^\perp(X) \in \mathcal{W} \subseteq \Omega \quad \text{and} \quad \tilde{\phi}_k^\perp(X) \in \mathcal{W} \subseteq \Omega, \quad k \in \mathbb{N}.$$

It follows that $\tilde{\phi}_k^\perp(\tilde{\psi}_k(X)) = \tilde{\psi}_k(\tilde{\phi}_k^\perp(X)) \in \Omega$ for each k . Let S be a weak* cluster point of $(\tilde{\phi}_k(\tilde{\psi}_k(X)))_{k \in \mathbb{N}}$. Since

$$\tilde{\phi}_k(\tilde{\psi}_k(X)) \in \overline{(\mathcal{M}_k \cap \text{Ran } \psi_k) \bar{\otimes} \mathcal{U} + \mathcal{W}}$$

for all $k \in \mathbb{N}$, we have, by Lemma 4.17, that S belongs to $\overline{\cap_k (\mathcal{M}_k \cap \text{Ran } \psi_k) \bar{\otimes} \mathcal{U} + \mathcal{W}}$, which is a subset of Ω . Since

$$X = \tilde{\phi}_k(\tilde{\psi}_k(X)) + \tilde{\phi}_k^\perp(\tilde{\psi}_k(X)) + \tilde{\theta}_k(X) + (\tilde{\psi}_k + \tilde{\theta}_k)^\perp(X), \quad k \in \mathbb{N},$$

it now follows that $X \in \Omega$. \square

Lemma 4.19. *Proposition 4.3 holds if the space \mathcal{W} has $r + 1$ terms.*

Proof. We use induction on the width n of \mathcal{C} . If $n = 1$ the statement reduces to Lemma 4.18. Suppose that the statement holds for all masa-bimodules \mathcal{C} of width not exceeding $n - 1$ and let $\mathcal{C} = \cap_{l=1}^n \mathcal{C}_l$, where \mathcal{C}_l is a nest algebra bimodule, $l = 1, \dots, n$. Fix

$$X \in \overline{\mathcal{M} \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{\mathcal{C} \bar{\otimes} \mathcal{U} + \mathcal{W}}$$

and let $(\psi_k, \theta_k, \text{Ran } \psi_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{C}_n . We also assume that $\mathcal{M} = \cap_k \mathcal{M}_k$ where every $(\mathcal{M}_k)_{k \in \mathbb{N}}$ is descending sequence of ranges of contractive Schur idempotents. Using Lemma 3.2 and the inductive assumption, we obtain

$$\begin{aligned} \tilde{\psi}_k^\perp(X) &\in \overline{\mathcal{M} \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{(\cap_{l=1}^{n-1} \mathcal{C}_l) \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{\mathcal{Z}_k \bar{\otimes} \mathcal{U} + \mathcal{B} \bar{\otimes} \mathcal{V}} \\ &= \overline{\mathcal{M} \cap (\cap_{l=1}^{n-1} \mathcal{C}_l) \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{\mathcal{Z}_k \bar{\otimes} \mathcal{U} + \mathcal{W}} \\ &= \overline{(\mathcal{M} \cap (\cap_{l=1}^{n-1} \mathcal{C}_l) \cap \mathcal{Z}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \overline{(\mathcal{M} \cap \mathcal{C}) \bar{\otimes} \mathcal{U} + \mathcal{W}}. \end{aligned}$$

Using Lemma 3.2 again, we have

$$\begin{aligned} \tilde{\psi}_k(X) &\in \overline{\mathcal{M}_k \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{(\cap_{l=1}^{n-1} \mathcal{C}_l) \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{\text{Ran } \psi_k \bar{\otimes} \mathcal{U} + \mathcal{W}} \\ &= \overline{(\mathcal{M}_k \cap \text{Ran } \psi_k) \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{(\cap_{l=1}^{n-1} \mathcal{C}_l) \bar{\otimes} \mathcal{U} + \mathcal{W}}, \end{aligned}$$

for every $k \in \mathbb{N}$. By Lemma 4.17, if S is a weak* cluster point of $(\tilde{\psi}_k(X))_{k \in \mathbb{N}}$, then

$$S \in \overline{(\mathcal{M} \cap \mathcal{N}) \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{(\cap_{l=1}^{n-1} \mathcal{C}_l) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

By the inductive assumption,

$$S \in \overline{(\mathcal{M} \cap \mathcal{N} \cap (\cap_{l=1}^{n-1} \mathcal{C}_l)) \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \overline{(\mathcal{M} \cap \mathcal{C}) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

The proof is complete. \square

Lemma 4.20. *Theorem 4.4 holds if the space \mathcal{W} has $r + 1$ terms.*

Proof. It suffices to prove that if $\mathcal{C}_1, \dots, \mathcal{C}_m$ are weak* closed nest algebra bimodules and $\mathcal{C} = \cap_{i=1}^m \mathcal{C}_i$, then

$$\cap_{i=1}^m \overline{\mathcal{C}_i \bar{\otimes} \mathcal{U} + \mathcal{W}} = \overline{\mathcal{C} \bar{\otimes} \mathcal{U} + \mathcal{W}},$$

where $\mathcal{C} = \cap_{i=1}^m \mathcal{C}_i$. We use induction on m . Suppose that

$$\cap_{i=1}^{m-1} \overline{\mathcal{C}_i \bar{\otimes} \mathcal{U} + \mathcal{W}} = \overline{(\cap_{i=1}^{m-1} \mathcal{C}_i) \bar{\otimes} \mathcal{U} + \mathcal{W}}$$

and fix $X \in \cap_{i=1}^m \overline{\mathcal{C}_i \bar{\otimes} \mathcal{U} + \mathcal{W}}$. Let $(\psi_k, \theta_k, \mathcal{M}_k, \mathcal{Z}_k)_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{C}_m . Using the inductive assumption and Lemmas 3.1 and 3.2, we have

$$\tilde{\psi}_k^\perp(X) \in \overline{((\cap_{i=1}^{m-1} \mathcal{C}_i) \cap \mathcal{Z}_k) \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \overline{\mathcal{C} \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

On the other hand,

$$\tilde{\psi}_k(X) \in \overline{\mathcal{M}_k \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{(\cap_{i=1}^{m-1} \mathcal{C}_i) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

Thus, if S is a weak* cluster point of the sequence $(\psi_k(X))_{k \in \mathbb{N}}$ then, by Lemma 4.17, we have that

$$S \in \overline{\mathcal{M} \bar{\otimes} \mathcal{U} + \mathcal{W}} \cap \overline{(\cap_{i=1}^{m-1} \mathcal{C}_i) \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

By Lemma 4.19,

$$S \in \overline{(\mathcal{M} \cap (\cap_{i=1}^{m-1} \mathcal{C}_i)) \bar{\otimes} \mathcal{U} + \mathcal{W}} \subseteq \overline{\mathcal{C} \bar{\otimes} \mathcal{U} + \mathcal{W}}.$$

The proof is complete. \square

Lemmas 4.5–4.20 conclude the proof of Propositions 4.1–4.3 and Theorem 4.4. The following statement, which is an equivalent formulation of Theorem 4.4, follows from that theorem by a straightforward induction on r .

Corollary 4.21. *Let $r, l_1, \dots, l_r \in \mathbb{N}$, $\{\mathcal{B}_j^i\}_{j=1}^{l_i}$ be a family of masa bimodules of finite width and \mathcal{U}_i be a weak* closed subspace of $\mathcal{B}(K)$, $i = 1, \dots, r$. Set $\mathcal{B}^i = \cap_{j=1}^{l_i} \mathcal{B}_j^i$, $i = 1, \dots, r$. Then*

$$\bigcap_{j_1, \dots, j_r} \overline{\mathcal{B}_{j_1}^1 \bar{\otimes} \mathcal{U}_1 + \dots + \mathcal{B}_{j_r}^r \bar{\otimes} \mathcal{U}_r} = \overline{\mathcal{B}^1 \bar{\otimes} \mathcal{U}_1 + \dots + \mathcal{B}^r \bar{\otimes} \mathcal{U}_r}.$$

5. OPERATOR SYNTHESIS OF UNIONS OF PRODUCTS

In this section we apply the results from Sections 3 and 4 to study questions about operator synthesis. We start by recalling the main definitions regarding the notion of operator synthesis.

Let (X, μ) be a standard measure space, that is, the measure μ is a regular Borel measure with respect to some Borel structures on X arising from a complete metrizable topology. Let $H = L^2(X, \mu)$. For a function $\varphi \in L^\infty(X, \mu)$, let M_φ be the (bounded) operator on H given by $M_\varphi f = \varphi f$, $f \in L^2(X, \mu)$. Let

$$\mathcal{D} = \{M_\varphi : \varphi \in L^\infty(X, \mu)\}.$$

We have that \mathcal{D} is a masa. We need several facts and notions from the theory of masa-bimodules [1], [9], [25]. A subset $E \subseteq X \times X$ is called *marginally null* if $E \subseteq (M \times X) \cup (X \times M)$, where $\mu(M) = 0$. We call two subsets $E, F \subseteq X \times X$ *marginally equivalent* (and write $E \cong F$) if the symmetric difference of E and F is marginally null. A set $\kappa \subseteq X \times X$ is called ω -open if it is marginally equivalent to a (countable) union of the form $\cup_{i=1}^\infty \alpha_i \times \beta_i$, where $\alpha_i \subseteq X$ and $\beta_i \subseteq X$ are measurable, $i \in \mathbb{N}$. The complements of ω -open sets are called ω -closed. An operator $T \in \mathcal{B}(H)$ is said to be supported on κ if $M_{\chi_\beta} T M_{\chi_\alpha} = 0$ whenever $(\alpha \times \beta) \cap \kappa \cong \emptyset$ (here χ_γ stands for the characteristic function of a measurable set γ). Given an ω -closed set $\kappa \subseteq X \times X$, let

$$\mathfrak{M}_{\max}(\kappa) = \{T \in \mathcal{B}(H) : T \text{ is supported on } \kappa\}.$$

The space $\mathfrak{M}_{\max}(\kappa)$ is a *reflexive* masa-bimodule in the sense that $\text{Ref}(\mathfrak{M}_{\max}(\kappa)) = \mathfrak{M}_{\max}(\kappa)$ where, for a subspace $\mathcal{U} \subseteq \mathcal{B}(H)$, we let its *reflexive hull* [19] be the subspace

$$\text{Ref}(\mathcal{U}) = \{T \in \mathcal{B}(H) : Tx \in \overline{\mathcal{U}x}, \text{ for all } x \in H\}.$$

We note two straightforward properties of the reflexive hull that will be used in the sequel: it is monotone ($\mathcal{U}_1 \subseteq \mathcal{U}_2$ implies $\text{Ref}(\mathcal{U}_1) \subseteq \text{Ref}(\mathcal{U}_2)$) and idempotent ($\text{Ref}(\text{Ref}(\mathcal{U})) = \text{Ref}(\mathcal{U})$).

It was shown in [9] that every reflexive masa-bimodule is of the form $\mathfrak{M}_{\max}(\kappa)$ for some, unique up to marginal equivalence, ω -closed set $\kappa \subseteq X \times X$. If \mathcal{U} is any masa-bimodule, then its *support* $\text{supp } \mathcal{U}$ is defined to be the ω -closed set $\kappa \subseteq X \times X$ such that $\text{Ref}(\mathcal{U}) = \mathfrak{M}_{\max}(\kappa)$. The masa-bimodule $\mathfrak{M}_{\max}(\kappa)$ is the largest, with respect to inclusion, (weak* closed) masa-bimodule with support κ (see [9]). As an extension of Arveson's work on commutative subspace lattices [1], it was shown in [25] that if κ is an ω -closed set, then there exists a smallest, with respect to inclusion, (weak* closed) masa-bimodule $\mathfrak{M}_{\min}(\kappa)$ with support κ . The ω -closed subset $\kappa \subseteq X \times X$ is called *operator synthetic* if $\mathfrak{M}_{\min}(\kappa) = \mathfrak{M}_{\max}(\kappa)$. The roots of the notion of operator synthesis lie in Harmonic Analysis – it is an operator theoretic version of the well-known concept of spectral synthesis. We refer the reader to [1] for a relevant discussion, and to [25] for the formal relation between the two concepts, which will be briefly summarised at the end of the section.

The supports of masa-bimodules of finite width will be called *sets of finite width*. A set $\kappa \subseteq X \times X$ is of finite width precisely when it is the set of solutions of a system of (finitely many) measurable function inequalities, that is, precisely when it has the form

$$\kappa = \{(x, y) \in X \times X : f_k(x) \leq g_k(y), k = 1, \dots, n\},$$

where $f_k : X \rightarrow \mathbb{R}$ and $g_k : X \rightarrow \mathbb{R}$ are measurable functions, $k = 1, \dots, n$ (see, e.g., [28]). It was shown in [25] and [28] that sets of finite width are operator synthetic.

In this section, we will be concerned with the question of when operator synthesis is preserved under unions of products. Suppose that (Y, ν) is another standard measure space, $K = L^2(Y, \nu)$ and \mathcal{C} is the multiplication masa of $L^\infty(Y, \nu)$. Let $\mathcal{U} \subseteq \mathcal{B}(H)$ be a \mathcal{D} -bimodule and $\mathcal{V} \subseteq \mathcal{B}(K)$ be a \mathcal{C} -bimodule. Then the subspace $\mathcal{U} \bar{\otimes} \mathcal{V}$ is a $\mathcal{D} \bar{\otimes} \mathcal{C}$ -bimodule, and hence its support is a subset of $(X \times Y) \times (X \times Y)$. The “flip”

$$\rho : (X \times X) \times (Y \times Y) \rightarrow (X \times Y) \times (X \times Y),$$

given by

$$\rho(x_1, x_2, y_1, y_2) = (x_1, y_1, x_2, y_2), \quad x_i \in X, y_i \in Y,$$

is thus needed in order to relate $\text{supp}(\mathcal{U} \bar{\otimes} \mathcal{V})$ to $(\text{supp } \mathcal{U}) \times (\text{supp } \mathcal{V})$. Indeed, it was shown in [21] that

$$(5) \quad \text{supp}(\mathcal{U} \bar{\otimes} \mathcal{V}) = \rho(\text{supp } \mathcal{U} \times \text{supp } \mathcal{V}).$$

It was observed in [24, Theorem 4.13] that

$$(6) \quad \mathfrak{M}_{\min}(\rho(\kappa \times \lambda)) = \mathfrak{M}_{\min}(\kappa) \bar{\otimes} \mathfrak{M}_{\min}(\lambda),$$

whenever $\kappa \subseteq X \times X$ and $\lambda \subseteq Y \times Y$ are ω -closed sets.

Remark 5.1. If $\kappa \subseteq X \times X$ and $\lambda \subseteq Y \times Y$ are non-marginally null ω -closed sets such that $\rho(\kappa \times \lambda)$ is operator synthetic, then both κ and λ are operator synthetic. Indeed, suppose that $T \in \mathfrak{M}_{\max}(\kappa)$, and let $0 \neq S \in \mathfrak{M}_{\min}(\lambda)$. Then $T \otimes S \in \mathfrak{M}_{\max}(\rho(\kappa \times \lambda))$ and, by assumption and identity (6),

$$T \otimes S \in \mathfrak{M}_{\min}(\kappa) \bar{\otimes} \mathfrak{M}_{\min}(\lambda).$$

It now easily follows that $T \in \mathfrak{M}_{\min}(\kappa)$. Thus, κ is operator synthetic and by symmetry λ is so as well.

Remark 5.2. Let G and H be locally compact groups. A problem in Harmonic Analysis asks when, given closed sets $E \subseteq G$ and $F \subseteq H$ satisfying spectral synthesis, the set $E \times F$ satisfies spectral synthesis as a subset of the direct product $G \times H$. We refer the reader to [11] for the definition of the notion of spectral synthesis and other basic concepts and results from non-commutative Harmonic Analysis. Analogues of identities (5) and (6) in the setting of Harmonic Analysis can be formulated as follows. Let $\text{VN}(G) \subseteq \mathcal{B}(L^2(G))$ (resp. $\text{VN}(H) \subseteq \mathcal{B}(L^2(H))$) be the von Neumann algebra of G (resp. H), and note that $\text{VN}(G) \bar{\otimes} \text{VN}(H)$ can be naturally identified with $\text{VN}(G \times H)$. The Harmonic Analysis analogue of masa-bimodules are *invariant spaces*; these are subspaces $\mathcal{X} \subseteq \text{VN}(G)$ that are annihilators of ideals of the Fourier algebra $A(G)$ of G . Given an invariant space $\mathcal{X} \subseteq \text{VN}(G)$, one may define its support $\text{supp } \mathcal{X}$ as the null set of its preannihilator in $A(G)$. It is not difficult to see that if $\mathcal{X} \subseteq \text{VN}(G)$ and $\mathcal{Y} \subseteq \text{VN}(H)$ are invariant spaces, then $\text{supp}(\mathcal{X} \bar{\otimes} \mathcal{Y}) = (\text{supp } \mathcal{X}) \times (\text{supp } \mathcal{Y})$ and that, given any closed subset $E \subseteq G$, there exists a largest (resp. smallest) invariant space $\mathcal{X}_{\max}(E)$ (resp. $\mathcal{X}_{\min}(E)$) with support E , and $\mathcal{X}_{\min}(E) \bar{\otimes} \mathcal{X}_{\min}(F) = \mathcal{X}_{\min}(E \times F)$.

The next proposition describes the connection between operator synthesis and tensor product formulas.

Proposition 5.3. *Let $\mathcal{U} \subseteq \mathcal{B}(H)$ and $\mathcal{V} \subseteq \mathcal{B}(K)$ be masa-bimodules with supports $\kappa \subseteq X \times X$ and $\lambda \subseteq Y \times Y$, respectively. Then*

$$(7) \quad \text{supp } \mathcal{F}(\mathcal{U}, \mathcal{V}) = \rho(\kappa \times \lambda)$$

and

$$(8) \quad \mathcal{F}(\mathfrak{M}_{\max}(\kappa), \mathfrak{M}_{\max}(\lambda)) = \mathfrak{M}_{\max}(\rho(\kappa \times \lambda)).$$

Moreover, if κ and λ are operator synthetic, then the following statements are equivalent:

- (i) $\rho(\kappa \times \lambda)$ is operator synthetic;
- (ii) $\mathcal{F}(\mathfrak{M}_{\max}(\kappa), \mathfrak{M}_{\max}(\lambda)) = \mathfrak{M}_{\max}(\kappa) \bar{\otimes} \mathfrak{M}_{\max}(\lambda)$;
- (iii) $\mathcal{F}(\mathfrak{M}_{\min}(\kappa), \mathfrak{M}_{\min}(\lambda)) = \mathfrak{M}_{\min}(\kappa) \bar{\otimes} \mathfrak{M}_{\min}(\lambda)$.

Proof. We have that $\mathcal{F}(\mathcal{U}, \mathcal{V}) = (\mathcal{U} \bar{\otimes} \mathcal{B}(K)) \cap (\mathcal{B}(H) \bar{\otimes} \mathcal{V})$, and hence

$$\text{supp } \mathcal{F}(\mathcal{U}, \mathcal{V}) = \text{supp}(\mathcal{U} \bar{\otimes} \mathcal{B}(K)) \cap \text{supp}(\mathcal{B}(H) \bar{\otimes} \mathcal{V}).$$

By (5), the support of $\mathcal{U} \bar{\otimes} \mathcal{B}(K)$ (resp. $\mathcal{B}(H) \bar{\otimes} \mathcal{V}$) is $\rho(\kappa \times (Y \times Y))$ (resp. $\rho((X \times X) \times \lambda)$). Identity (7) now readily follows. To establish (8) note that $\mathcal{F}(\mathfrak{M}_{\max}(\kappa), \mathfrak{M}_{\max}(\lambda))$ and $\mathfrak{M}_{\max}(\rho(\kappa \times \lambda))$ are both reflexive and, by (7), have equal supports.

Suppose that κ and λ are operator synthetic.

(ii) \Leftrightarrow (i) Using [24, Theorem 4.13] for the first equality below and identity (8) for the last one, we have

$$\begin{aligned} \mathfrak{M}_{\min}(\rho(\kappa \times \lambda)) &= \mathfrak{M}_{\min}(\kappa) \bar{\otimes} \mathfrak{M}_{\min}(\lambda) = \mathfrak{M}_{\max}(\kappa) \bar{\otimes} \mathfrak{M}_{\max}(\lambda) \\ &\subseteq \mathcal{F}(\mathfrak{M}_{\max}(\kappa), \mathfrak{M}_{\max}(\lambda)) = \mathfrak{M}_{\max}(\rho(\kappa \times \lambda)). \end{aligned}$$

If the inclusion in the above chain is equality then we have that $\mathfrak{M}_{\min}(\rho(\kappa \times \lambda)) = \mathfrak{M}_{\max}(\rho(\kappa \times \lambda))$, in other words, that $\rho(\kappa \times \lambda)$ is operator synthetic. Conversely, if $\rho(\kappa \times \lambda)$ is operator synthetic then we must have equalities throughout.

(iii) \Leftrightarrow (i) follows similarly from the chain

$$\begin{aligned} \mathfrak{M}_{\min}(\rho(\kappa \times \lambda)) &= \mathfrak{M}_{\min}(\kappa) \bar{\otimes} \mathfrak{M}_{\min}(\lambda) \subseteq \mathcal{F}(\mathfrak{M}_{\min}(\kappa), \mathfrak{M}_{\min}(\lambda)) \\ &= (\mathfrak{M}_{\min}(\kappa) \bar{\otimes} \mathcal{B}(K_1, K_2)) \cap (\mathcal{B}(H_1, H_2) \bar{\otimes} \mathfrak{M}_{\min}(\lambda)) \\ &= (\mathfrak{M}_{\max}(\kappa) \bar{\otimes} \mathcal{B}(K_1, K_2)) \cap (\mathcal{B}(H_1, H_2) \bar{\otimes} \mathfrak{M}_{\max}(\lambda)) \\ &= \mathcal{F}(\mathfrak{M}_{\max}(\kappa), \mathfrak{M}_{\max}(\lambda)) = \mathfrak{M}_{\max}(\rho(\kappa \times \lambda)). \end{aligned}$$

□

Corollary 5.4. *Let $\kappa \subseteq X \times X$ be an operator synthetic ω -closed set. If $\mathfrak{M}_{\max}(\kappa)$ has property S_σ then $\rho(\kappa \times \lambda)$ is operator synthetic for every operator synthetic ω -closed set $\lambda \subseteq Y \times Y$.*

Proof. Immediate from Proposition 5.3 (ii) \Leftrightarrow (i). □

It follows from Corollary 5.4 that if κ is a set of finite width then $\rho(\kappa \times \lambda)$ is operator synthetic whenever λ is so. In fact, we have the following stronger result.

Corollary 5.5. *Let $\kappa \subseteq X \times X$ and $\lambda \subseteq Y \times Y$ be operator synthetic sets and $\kappa' \subseteq X \times X$ be an ω -closed set of finite width. If $\rho(\kappa \times \lambda)$ is operator synthetic then so is $\rho((\kappa \cup \kappa') \times \lambda)$.*

Proof. Let $\mathcal{V} = \mathfrak{M}_{\max}(\kappa)$, $\mathcal{B} = \mathfrak{M}_{\max}(\kappa')$ and $\mathcal{U} = \mathfrak{M}_{\max}(\lambda)$. It is straightforward to check that the support of $\overline{\mathcal{V} + \mathcal{B}}$ is $\kappa \cup \kappa'$. By [8, Corollary 4.2], $\kappa \cup \kappa'$ is operator synthetic, and hence $\mathfrak{M}_{\max}(\kappa \cup \kappa') = \overline{\mathcal{V} + \mathcal{B}}$. By Proposition 5.3,

$$\mathcal{F}(\mathfrak{M}_{\max}(\kappa), \mathfrak{M}_{\max}(\lambda)) = \mathfrak{M}_{\max}(\kappa) \bar{\otimes} \mathfrak{M}_{\max}(\lambda).$$

By Theorem 3.7,

$$\mathcal{F}(\mathfrak{M}_{\max}(\kappa \cup \kappa'), \mathfrak{M}_{\max}(\lambda)) = \mathfrak{M}_{\max}(\kappa \cup \kappa') \bar{\otimes} \mathfrak{M}_{\max}(\lambda).$$

By Proposition 5.3, $\rho((\kappa \cup \kappa') \times \lambda)$ is operator synthetic. □

Our next aim is Theorem 5.9, for whose proof we will need some auxiliary lemmas.

Lemma 5.6. *Let $\mathcal{U} \subseteq \mathcal{B}(H)$ be a masa-bimodule and ϕ be a Schur idempotent acting on $\mathcal{B}(H)$. Then $\phi(\text{Ref}(\mathcal{U})) = \text{Ref}(\phi(\mathcal{U})) = \text{Ran } \phi \cap \text{Ref}(\mathcal{U})$.*

Proof. By [8, Proposition 3.3], $\text{Ref}(\mathcal{U})$ coincides with the space of all operators $X \in \mathcal{B}(H)$ such that $\psi(X) = 0$ whenever ψ is a Schur idempotent annihilating \mathcal{U} . Fix $T \in \text{Ref}(\mathcal{U})$ and let θ be a Schur idempotent on $\mathcal{B}(H)$ such that $\theta(\phi(\mathcal{U})) = \{0\}$. Then $\theta \circ \phi(T) = 0$ and hence $\phi(T) \in \text{Ref}(\phi(\mathcal{U}))$; we thus showed that $\phi(\text{Ref}(\mathcal{U})) \subseteq \text{Ref}(\phi(\mathcal{U}))$.

Now suppose that $T \in \phi(\text{Ref}(\mathcal{U}))$; then clearly $T \in \text{Ran } \phi$ and, by the previous paragraph, $T \in \text{Ref}(\phi(\mathcal{U})) \subseteq \text{Ref}(\mathcal{U})$. Thus, $\phi(\text{Ref}(\mathcal{U})) \subseteq \text{Ran } \phi \cap \text{Ref}(\mathcal{U})$. On the other hand, if $T \in \text{Ran } \phi \cap \text{Ref}(\mathcal{U})$ then $T = \phi(T) \in \phi(\text{Ref}(\mathcal{U}))$; hence, $\phi(\text{Ref}(\mathcal{U})) = \text{Ran } \phi \cap \text{Ref}(\mathcal{U})$.

By [8, Proposition 3.3], $\text{Ran } \phi$ is reflexive and since reflexivity is preserved by intersections, the previous paragraph implies that $\phi(\text{Ref}(\mathcal{U}))$ is reflexive. Since $\phi(\mathcal{U}) \subseteq \phi(\text{Ref}(\mathcal{U}))$, we have $\text{Ref}(\phi(\mathcal{U})) \subseteq \text{Ref}(\phi(\text{Ref}(\mathcal{U}))) = \phi(\text{Ref}(\mathcal{U}))$, and the proof is complete. \square

Lemma 5.7. *Let ϕ_i be a Schur idempotent, $\kappa_i \subseteq X \times X$ be the support of $\text{Ran } \phi_i$, and $\lambda_i \subseteq Y \times Y$ be an ω -closed set, $i = 1, \dots, r$. Suppose that $\cup_{k=1}^p \lambda_{m_k}$ is operator synthetic, whenever $1 \leq m_1 < m_2 < \dots < m_p \leq r$. Then the set $\rho(\cup_{i=1}^r \kappa_i \times \lambda_i)$ is operator synthetic.*

Proof. Set $\kappa = \rho(\cup_{i=1}^r \kappa_i \times \lambda_i)$, $\mathcal{U}_i = \mathfrak{M}_{\min}(\lambda_i)$ and $\mathcal{W} = \mathfrak{M}_{\min}(\kappa)$. By (5), the support of the masa-bimodule $\overline{\sum_{i=1}^r \mathfrak{M}_{\min}(\kappa_i) \bar{\otimes} \mathfrak{M}_{\min}(\lambda_i)}$ is $\rho(\cup_{i=1}^r \kappa_i \times \lambda_i)$; by the minimality property of \mathcal{W} and the fact that the sets κ_i and λ_i are operator synthetic, we have that

$$\mathcal{W} = \overline{\sum_{i=1}^r \mathfrak{M}_{\min}(\kappa_i) \bar{\otimes} \mathfrak{M}_{\min}(\lambda_i)} = \overline{\sum_{i=1}^r \mathfrak{M}_{\max}(\kappa_i) \bar{\otimes} \mathfrak{M}_{\max}(\lambda_i)}.$$

For each $i = 1, \dots, r$, let $\phi_i^1 = \phi_i$ and $\phi_i^{-1} = \phi_i^\perp$, and for each subset M of $\{1, \dots, r\}$, let $\phi_M = \phi_1^{\epsilon_1} \phi_2^{\epsilon_2} \dots \phi_r^{\epsilon_r}$, where $\epsilon_i = 1$ if $i \in M$ and $\epsilon_i = -1$ if $i \notin M$.

Fix $T \in \text{Ref}(\mathcal{W})$; we will show that $T \in \mathcal{W}$. This will then imply that $\mathcal{W} = \text{Ref}(\mathcal{W})$, and hence that $\rho(\cup_{i=1}^r \kappa_i \times \lambda_i)$ is operator synthetic.

We have $T = \sum \tilde{\phi}_M(T)$, where the sum is taken over all subsets M of $\{1, \dots, r\}$. By Lemma 5.6, $\tilde{\phi}_M(T) \in \text{Ref}(\tilde{\phi}_M(\mathcal{W}))$ and hence

$$T \in \text{Ref} \left(\sum_M \sum_{i=1}^r \tilde{\phi}_M(\text{Ran } \phi_i \bar{\otimes} \mathcal{U}_i) \right).$$

By Lemma 3.1, $\tilde{\phi}_M(\text{Ran } \phi_i \bar{\otimes} \mathcal{U}_i) = \text{Ran } \phi_M \bar{\otimes} \mathcal{U}_i$ if $i \in M$, and $\tilde{\phi}_M(\text{Ran } \phi_i \bar{\otimes} \mathcal{U}_i) = \{0\}$ otherwise. Thus,

$$T \in \text{Ref} \sum_M \left(\text{Ran } \phi_M \bar{\otimes} \overline{\sum_{i \in M} \mathcal{U}_i} \right).$$

We have that $\phi_M \phi_N = 0$ if $M \neq N$. The assumption concerning the synthesis of the finite unions of the sets λ_j implies that $\overline{\sum_{i \in M} \mathcal{U}_i} = \mathfrak{M}_{\max}(\cup_{i \in M} \lambda_i)$; we may thus assume that the maps ϕ_i , $i = 1, \dots, r$ have the property that $\phi_i \phi_j = 0$ if $i \neq j$.

We now proceed by induction on r . If $r = 1$, the statement follows from Lemma 3.1 and Corollary 5.4. Assume that the statement holds if the number of the given terms is at most $r - 1$, and recall that $T \in \text{Ref}(\mathcal{W})$. By Lemma 5.6 and the inductive assumption,

$$\tilde{\phi}_r^\perp(T) \in \text{Ref}(\tilde{\phi}_r^\perp(\mathcal{W})) \subseteq \text{Ref} \left(\sum_{i=1}^{r-1} \mathfrak{M}_{\max}(\kappa_i) \bar{\otimes} \mathfrak{M}_{\max}(\lambda_i) \right) \subseteq \mathcal{W}.$$

On the other hand,

$$\begin{aligned} \tilde{\phi}_r(T) &\in \text{Ref}(\mathfrak{M}_{\max}(\kappa_r) \bar{\otimes} \mathfrak{M}_{\max}(\lambda_r)) = \mathfrak{M}_{\max}(\kappa_r) \bar{\otimes} \mathfrak{M}_{\max}(\lambda_r) \\ &= \mathfrak{M}_{\min}(\kappa_r) \bar{\otimes} \mathfrak{M}_{\min}(\lambda_r) \subseteq \mathcal{W} \end{aligned}$$

(we have used Lemma 5.6 for the containment, and Corollary 5.4, Proposition 5.3 and the fact that $\mathfrak{M}_{\max}(\kappa_r)$ has property S_σ for the first equality). Thus,

$$T = \tilde{\phi}_r(T) + \tilde{\phi}_r^\perp(T) \in \mathcal{W}$$

and the proof is complete. \square

Lemma 5.8. *Let $\kappa_i \subseteq X \times X$ be the support of a nest algebra bimodule, and let $\lambda_i \subseteq Y \times Y$ be an ω -closed set, $i = 1, \dots, r$. Suppose that $\cup_{k=1}^p \lambda_{m_k}$ is operator synthetic whenever $1 \leq m_1 < m_2 < \dots < m_p \leq r$. Then the set $\rho(\cup_{i=1}^r \kappa_i \times \lambda_i)$ is operator synthetic.*

Proof. Set $\kappa = \rho(\cup_{i=1}^r \kappa_i \times \lambda_i)$. Let $\mathcal{B}_i = \mathfrak{M}_{\max}(\kappa_i)$, $\mathcal{U}_i = \mathfrak{M}_{\max}(\lambda_i)$, $i = 1, \dots, r$, and $\mathcal{W} = \overline{\sum_{i=1}^r \mathcal{B}_i \bar{\otimes} \mathcal{U}_i}$. As in the proof of Lemma 5.7, $\mathcal{W} = \mathfrak{M}_{\min}(\kappa)$ and hence $\text{Ref}(\mathcal{W}) = \mathfrak{M}_{\max}(\kappa)$.

Let $(\phi_{i,k}, \theta_{i,k}, \mathcal{M}_{i,k}, \mathcal{Z}_{i,k})_{k \in \mathbb{N}}$ be a decomposition scheme for \mathcal{B}_i , $i = 1, \dots, r$. Set $\psi_{i,k} = (\phi_{i,k} + \theta_{i,k})^\perp$. For each subset M of $\{1, \dots, r\}$, a subset N of M , and indices $k_1, k_2, \dots, k_r \in \mathbb{N}$, we let $\gamma_{k_1, k_2, \dots, k_r}^{M, N} = \gamma_1 \circ \dots \circ \gamma_r$, where $\gamma_i = \tilde{\phi}_{i, k_i}$ if $i \in N$, $\gamma_i = \tilde{\theta}_{i, k_i}$ if $i \in M \setminus N$ and $\gamma_i = \psi_{i, k_i}$ if $i \notin M$. Fix $T \in \text{Ref}(\mathcal{W})$. Then, by Lemmas 3.1 and 5.6,

$$\gamma_{k_1, k_2, \dots, k_r}^{M, N}(T) \in \text{Ref} \left(\sum_{i=1}^r \mathcal{Y}_{i, k_i} \bar{\otimes} \mathcal{U}_i \right),$$

where \mathcal{Y}_{i, k_i} is equal to \mathcal{M}_{i, k_i} if $i \in N$, to \mathcal{Z}_{i, k_i} if $i \in M \setminus N$ and to $\{0\}$ if $i \notin M$.

Moreover, for all k_1, k_2, \dots, k_r , we have that

$$T = \sum_{M, N} \gamma_{k_1, k_2, \dots, k_r}^{M, N}(T),$$

where the sum is taken over all subsets M and N of $\{1, \dots, n\}$ with $N \subseteq M$. By Lemma 5.7,

$$\gamma_{k_1, k_2, \dots, k_r}^{M, N}(T) \in \overline{\sum_{i=1}^r \mathcal{Y}_{i, k_i} \bar{\otimes} \mathcal{U}_i}.$$

Since $\mathcal{Z}_{i, k} \subseteq \mathcal{B}_i$ for every $k \in \mathbb{N}$, we have that

$$\gamma_{k_1, k_2, \dots, k_r}^{M, N}(T) \in \overline{\sum_{i=1}^r \mathcal{X}_{i, k_i} \bar{\otimes} \mathcal{U}_i},$$

where \mathcal{X}_{i, k_i} is equal to \mathcal{M}_{i, k_i} if $i \in N$, to \mathcal{B}_i if $i \in M \setminus N$ and to $\{0\}$ if $i \notin M$.

Let $\{(M_p, N_p)\}_{p=1}^q$ be an enumeration of the pairs of sets (M, N) with $N \subseteq M \subseteq \{1, \dots, r\}$. For every fixed $r-1$ -tuple (k_1, \dots, k_{r-1}) of indices, choose a weak* convergent subsequence $(\gamma_{k_1, k_2, \dots, k_{r-1}, k'_r}^{M_1, N_1}(T))_{k'_r \in \mathbb{N}}$ of the sequence $(\gamma_{k_1, k_2, \dots, k_r}^{M_1, N_1}(T))_{k_r \in \mathbb{N}}$, and let $\gamma_{k_1, k_2, \dots, k_{r-1}}^{M_1, N_1}(T)$ be its limit. Then choose a weak* convergent subsequence $(\gamma_{k_1, k_2, \dots, k_{r-1}, k''_r}^{M_2, N_2}(T))_{k''_r \in \mathbb{N}}$ of the sequence $(\gamma_{k_1, k_2, \dots, k_{r-1}, k'_r}^{M_1, N_1}(T))_{k'_r \in \mathbb{N}}$, and let

$\gamma_{k_1, k_2, \dots, k_{r-1}}^{M_2, N_2}(T)$ be its limit. Continuing inductively, define, for each pair (M, N) , an operator $\gamma_{k_1, k_2, \dots, k_{r-1}}^{M, N}(T)$; by the choice of these operators, we have that

$$T = \sum_{M, N} \gamma_{k_1, k_2, \dots, k_{r-1}}^{M, N}(T).$$

By Proposition 4.1 (v),

$$\gamma_{k_1, k_2, \dots, k_{r-1}}^{M, N}(T) \in \overline{\sum_{i=1}^{r-1} \mathcal{X}_{i, k_i} \bar{\otimes} \mathcal{U}_i + \mathcal{B}_r \bar{\otimes} \mathcal{U}_r}.$$

We now choose, as in the previous paragraph, for every $r-2$ -tuple (k_1, \dots, k_{r-2}) of indices, a weak* cluster point $\gamma_{k_1, k_2, \dots, k_{r-2}}^{M, N}(T)$ of

$$(\gamma_{k_1, k_2, \dots, k_{r-1}}^{M, N}(T))_{k_{r-1} \in \mathbb{N}}$$

such that $T = \sum_{M, N} \gamma_{k_1, k_2, \dots, k_{r-2}}^{M, N}(T)$, and use Proposition 4.1 (v) to conclude that

$$\gamma_{k_1, k_2, \dots, k_{r-2}}^{M, N}(T) \in \overline{\sum_{i=1}^{r-2} \mathcal{X}_{i, k_i} \bar{\otimes} \mathcal{U}_i + \mathcal{B}_{r-1} \bar{\otimes} \mathcal{U}_{r-1} + \mathcal{B}_r \bar{\otimes} \mathcal{U}_r}.$$

Continuing inductively, we conclude that $T = \sum_{M, N} \gamma_{\emptyset}^{M, N}(T)$, where $\gamma_{\emptyset}^{M, N}(T) \in \mathcal{W}$ for all subsets N and M of $\{1, \dots, r\}$ with $N \subseteq M$. Hence, $T \in \mathcal{W}$ and the proof is complete. \square

Theorem 5.9. *Let $\kappa_i \subseteq X \times X$ be a set of finite width, and let $\lambda_i \subseteq Y \times Y$ be an ω -closed set, $i = 1, \dots, r$. Suppose that $\cup_{k=1}^p \lambda_{m_k}$ is operator synthetic whenever $1 \leq m_1 < m_2 < \dots < m_p \leq r$. Then the set $\rho(\cup_{i=1}^r \kappa_i \times \lambda_i)$ is operator synthetic.*

Proof. Let $\kappa = \rho(\cup_{i=1}^r \kappa_i \times \lambda_i)$, $\mathcal{B}_i = \mathfrak{M}_{\max}(\kappa_i)$ and write $\mathcal{B}_i = \cap_{j=1}^{l_i} \mathcal{B}_j^i$, where \mathcal{B}_j^i is a nest algebra bimodule, $i = 1, \dots, r$, $j = 1, \dots, l_i$. Let also $\mathcal{U}_i = \mathfrak{M}_{\max}(\lambda_i)$, $i = 1, \dots, r$. Fix

$$T \in \mathfrak{M}_{\max}(\kappa) = \text{Ref} \left(\sum_{i=1}^r \mathcal{B}_i \bar{\otimes} \mathcal{U}_i \right).$$

Lemma 5.8 implies that, for all j_1, \dots, j_r , we have

$$T \in \text{Ref} \left(\sum_{i=1}^r \mathcal{B}_{j_i}^i \bar{\otimes} \mathcal{U}_i \right) = \overline{\sum_{i=1}^r \mathcal{B}_{j_i}^i \bar{\otimes} \mathcal{U}_i}.$$

By Corollary 4.21,

$$T \in \overline{\sum_{i=1}^r \mathcal{B}_i \bar{\otimes} \mathcal{U}_i} = \mathfrak{M}_{\min}(\kappa).$$

\square

Remark 5.10. In Theorem 5.9, the condition that $\cup_{k=1}^p \lambda_{m_k}$ be operator synthetic whenever $1 \leq m_1 < m_2 < \dots < m_p \leq r$ cannot be omitted. Indeed, given such a choice of indices, fix a non-trivial subset of finite width κ , and let $\kappa_{m_j} = \kappa$, $j = 1, \dots, p$, and $\kappa_i = \emptyset$ if $i \notin \{m_1, \dots, m_p\}$. If $\rho(\cup_{i=1}^r \kappa_i \times \lambda_i) = \rho(\kappa \times (\cup_{j=1}^p \lambda_{m_j}))$ is operator synthetic then, by Remark 5.1, $\cup_{j=1}^p \lambda_{m_j}$ is operator synthetic.

We conclude this section with an application of the previous results to spectral synthesis. Let G be a second countable locally compact group. By [20], a closed set $E \subseteq G$ satisfies local spectral synthesis if and only if the set

$$E^* = \{(s, t) \in G \times G : st^{-1} \in E\}$$

is operator synthetic (here G is equipped with left Haar measure). We note that, in the case the Fourier algebra $A(G)$ has an approximate identity, E is of local spectral synthesis if and only if it satisfies spectral synthesis (see [20]).

Let \mathbb{R}^+ be the group of positive real numbers and $\omega : G \rightarrow \mathbb{R}^+$ be a continuous group homomorphism. For each $t > 0$, let

$$E_\omega^t = \{x \in G : \omega(x) \leq t\};$$

it is natural to call such a subset a *level set*. We have that

$$(E_\omega^t)^* = \{(x, y) \in G \times G : \omega(x) \leq t\omega(y)\}$$

and hence the intersections of the form

$$E = E_{\omega_1}^{t_1} \cap \dots \cap E_{\omega_k}^{t_k}$$

are a Harmonic Analysis version of sets of finite width: they have the property that the corresponding set E^* is a set of finite width (see also [8]). Theorem 5.9 has the following immediate consequence.

Corollary 5.11. *Let G and H be second countable locally compact groups. Suppose that E_1, \dots, E_r are level sets in G and F_1, \dots, F_r are closed subsets of H such that $\cup_{k=1}^p F_{m_k}$ is a set of local spectral synthesis whenever $1 \leq m_1 < m_2 < \dots < m_p \leq r$. Then the set $\cup_{i=1}^r E_i \times F_i$ is a set of local spectral synthesis of $G \times H$.*

6. FUBINI PRODUCTS AND MORITA EQUIVALENCE

In this section, we show how tensor product formulas relate to the notion of spacial Morita equivalence introduced in [6]. For subspaces \mathcal{X} and \mathcal{Y} of operators, we let

$$[\mathcal{X}\mathcal{Y}] = \overline{\left\{ \sum_{i=1}^k X_i Y_i : X_i \in \mathcal{X}, Y_i \in \mathcal{Y}, i = 1, \dots, k, k \in \mathbb{N} \right\}}.$$

We recall the following definition from [7]:

Definition 6.1. *Let \mathcal{A} (resp. \mathcal{B}) be a weak* closed unital algebra acting on a Hilbert space H_1 (resp. H_2). We say that \mathcal{A} is spatially embedded in \mathcal{B} if there exist a \mathcal{B}, \mathcal{A} -bimodule $\mathcal{X} \subseteq \mathcal{B}(H_1, H_2)$ and an \mathcal{A}, \mathcal{B} -bimodule $\mathcal{Y} \subseteq \mathcal{B}(H_2, H_1)$ such that $[\mathcal{X}\mathcal{Y}] \subseteq \mathcal{B}$ and $[\mathcal{Y}\mathcal{X}] = \mathcal{A}$.*

If, moreover, $\mathcal{B} = [\mathcal{X}\mathcal{Y}]$, we call \mathcal{A} and \mathcal{B} spatially Morita equivalent.

We note that if two unital dual operator algebras \mathcal{A} and \mathcal{B} are weak* Morita equivalent in the sense of [2] then they have completely isometric representations α and β such that the algebras $\alpha(\mathcal{A})$ and $\beta(\mathcal{B})$ are spatially Morita equivalent (this fact will not be used in the sequel).

Theorem 6.1. *Let H_1, H_2 and K be Hilbert spaces and $\mathcal{A} \subseteq \mathcal{B}(H_1)$ and $\mathcal{B} \subseteq \mathcal{B}(H_2)$ be weak* closed unital algebras. Suppose that \mathcal{A} is spatially embedded in \mathcal{B} and let $\mathcal{U} \subseteq \mathcal{B}(K)$ be a weak* closed space such that $\mathcal{F}(\mathcal{B}, \mathcal{U}) = \mathcal{B} \bar{\otimes} \mathcal{U}$. Then $\mathcal{F}(\mathcal{A}, \mathcal{U}) = \mathcal{A} \bar{\otimes} \mathcal{U}$.*

Proof. Let $\mathcal{X} \subseteq \mathcal{B}(H_1, H_2)$ and $\mathcal{Y} \subseteq \mathcal{B}(H_2, H_1)$ be subspaces satisfying the conditions of Definition 6.1, and $T \in \mathcal{A} \bar{\otimes} \mathcal{B}(K)$ be such that $R_\phi(T) \in \mathcal{U}$ for all $\phi \in \mathcal{B}(H_1, H_2)_*$. Suppose that

$$T = w^* - \lim_n \sum_{i=1}^{m_n} A_i^n \otimes S_i^n,$$

where $(A_i^n)_{i=1}^{m_n} \subseteq \mathcal{A}$ and $(S_i^n)_{i=1}^{m_n} \subseteq \mathcal{B}(K)$. Fix $X_1 \in \mathcal{X}$ and $Y_1 \in \mathcal{Y}$ and set $S = (X_1 \otimes I)T(Y_1 \otimes I)$. For $\psi \in \mathcal{B}(H_2)_*$, let $\phi \in \mathcal{B}(H_1)_*$ be given by $\phi(A) = \psi(X_1 A Y_1)$, $A \in \mathcal{B}(H_1)$. Since

$$R_\phi(T) = w^* - \lim_n \sum_{i=1}^{m_n} \phi(A_i^n) S_i^n \in \mathcal{U},$$

we have that $R_\psi(S) \in \mathcal{U}$, for every $\psi \in \mathcal{B}(H_2)_*$. By our assumption, $S \in \mathcal{B} \bar{\otimes} \mathcal{U}$. Therefore, $(X_1 \otimes I)T(Y_1 \otimes I) \in \mathcal{B} \bar{\otimes} \mathcal{U}$ for all $X_1 \in \mathcal{X}$ and all $Y_1 \in \mathcal{Y}$. It follows that

$$(Y_2 X_1 \otimes I)T(Y_1 X_2 \otimes I) \in \mathcal{A} \bar{\otimes} \mathcal{U}, \quad \text{for all } X_1, X_2 \in \mathcal{X}, Y_1, Y_2 \in \mathcal{Y}.$$

Since $I \in \mathcal{A} = [\mathcal{Y}\mathcal{X}]$, it follows that $T \in \mathcal{A} \bar{\otimes} \mathcal{U}$. \square

The following corollary is straightforward from Theorem 6.1.

Corollary 6.2. *Suppose that \mathcal{A} and \mathcal{B} are weak* closed unital operator algebras.*

(i) *Suppose that \mathcal{A} is spatially embedded in \mathcal{B} . If \mathcal{B} has property S_σ then so does \mathcal{A} .*

(ii) *Suppose that \mathcal{A} and \mathcal{B} are spatially Morita equivalent. Then \mathcal{A} has property S_σ precisely when \mathcal{B} does so.*

The last corollary, which is immediate from [7] and Corollary 6.2, concerns the inheritance of property S_σ in the class of CSL algebras; we refer the reader to [1] for the definition, relevant notation and theory of this class of algebras.

Corollary 6.3. *Let \mathcal{L}_1 and \mathcal{L}_2 be CSL's.*

(i) *Suppose that $\phi : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ is a strongly continuous surjective lattice homomorphism. If $\text{Alg}(\mathcal{L}_1)$ has property S_σ then so does $\text{Alg}(\mathcal{L}_2)$.*

(ii) *Suppose that $\phi : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ is a strongly continuous lattice isomorphism. Then the algebra $\text{Alg}(\mathcal{L}_1)$ has property S_σ if and only if $\text{Alg}(\mathcal{L}_2)$ does so.*

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